

## **FRAMES, FRACTALS AND RADIAL OPERATORS IN HILBERT SPACE**

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### **Abstract**

In this paper, we construct graphs having a fractal property, and groupoids induced by the graphs. The fundamental properties of them and their corresponding graph von Neumann algebras, and their radial operators are studied.

### **1. Introduction**

We are concerned with countable directed graphs  $G$  as they act on measure spaces. Our motivation comes in part from the study of weighted graphs used in infinite (very large) electric networks, in statistical models from physics, stochastic processes indexed by some fixed directed graph, or in the study renormalization and of fractals. These applications involve some similarity of scales. This in turn dictates a particular special

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interaction between the particular graph  $G$  under consideration, and its environment  $X$ . The latter will be modelled on a measure space, a framed system; see detailed definitions below.

**1.1. Overview.** Our aim is to identify certain similarity scales and renormalization structures in graphs. They will allow us to connect global aspects of a system  $G, X$  with its local parts, similarity of scales in the small and in the large. While this may be done in a variety of ways in general, we will focus here on one such approach, made precise with our use of radial operators for the system under consideration.

Our motivation comes in part from operator algebras, more specifically, from the theory of one-parameter groups of automorphisms acting on a von Neumann algebra. Our questions have a measure-theoretic flavor, which favors the use of von Neumann algebras, as opposed to  $C^*$ -algebras.

A *graph* is a set of objects called *vertices* (or points or nodes) connected by links called *edges* (or lines). In a *directed graph*, the two directions are counted as being distinct directed edges (or arcs). A graph is depicted in a diagrammatic form as a set of dots (for vertices), joined by curves (for edges). Similarly, a directed graph is depicted in a diagrammatic form as a set of dots joined by arrowed curves, where the arrows point in the direction of the directed edges. We are interested in countable directed graphs.

More precisely, a directed graph  $G$  is a pair  $(V(G), E(G))$ , with direction on  $E(G)$ , where  $V(G)$  is the *vertex set*, consisting of all vertices of  $G$ , and  $E(G)$  is the *edge set*, consisting of all directed edges of  $G$ . The direction on  $G$  creates the initial vertices and the terminal vertices of edges.

The algebraic structures, induced by directed graphs, have been studied recently (e.g., see [3] through [18]). In particular, in [3], the graph groupoids are defined by the groupoids induced by graphs (also, see [22]

and [38]). Depending on such groupoidal structures, we could construct von Neumann algebras preserving the combinatorial properties of graphs, and depending on the algebraic reduction of graph groupoids: We realized that, under the suitable representation, graph groupoids are nicely embedded in an operator algebra, and they generate the groupoid ( $W^*$ - or  $C^*$ -)dynamical system in the operator algebra. Thus, by the *groupoid crossed product*, we defined *graph von Neumann algebras* in [3], [4], and we characterized  $C^*$ -algebras generated by certain operators in [14], [17], and [18].

**1.2. Motivation.** The main purpose of this paper is to introduce new algebraic structures having certain fractal property, which is called *fractality*. In [15], we introduced these structures, called the *fractaloids*. Fractaloids in the sense of [15] are the *graph groupoids*, in the sense of Subsection 3.2 (below), satisfying certain additional conditions. In [12], we call the fractaloids, the *graph fractaloids*, to emphasize that they are special “graph” groupoids.

In [14], we conjectured that the connected “finite” directed graphs, generating graph fractaloids, are:

- (i) the one-vertex-multi-loop-edge graphs  $O_n$ , with

$$V(O_n) = \{v_O\},$$

and

$$E(O_n) = \{e_1, \dots, e_n\},$$

where  $e_j$  are the directed edges connecting  $v_O$  to  $v_O$ , for all  $n \in \mathbb{N}$ . For instance,



is the one-vertex-one-loop-edge graph, or

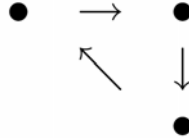
- (ii) the one-flow circulant graphs  $K_n$ , with

$$V(K_n) = \{v_1, \dots, v_n\},$$

and

$$E(K_n) = \{e_{12}, e_{23}, e_{34}, \dots, e_{n-1,n}, e_{n,1}\},$$

where  $e_{ij}$  means the edge connecting the vertex  $v_i$  to the vertex  $v_j$ , for  $n \in \mathbb{N} \setminus \{1\}$ . For instance,



is the one-flow circulant graph  $K_3$ , or

(iii) a suitable connection or unions of the graphs in (i) and (ii). And this conjecture is concluded in [9]. We realized that there are many more connected finite directed graphs generating graph fractaloids. Indeed, the conclusion of [9] shows the richness of fractaloids: There are sufficiently many “finite” fractal graphs (and hence there are sufficiently many fractal graphs).

Futhermore, in [10] and [19], we showed that, for any pair  $(n, m) \in \mathbb{N} \times \mathbb{N}_\infty$ , where  $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ , there exists at least one corresponding fractal graph, i.e., if  $\mathcal{F}_{ractal}$  is the collection of all connected locally finite fractal graphs, then it is decomposed by the equivalence classes  $[(n, m)]$ , called the *spectral classes*, by

$$\mathcal{F}_{ractal} = \bigsqcup_{(n,m) \in \mathbb{N} \times \mathbb{N}_\infty} [(n, m)]$$

(also, see below Section 3).

Independently, the measure framing on graphs have been studied by the authors (see [42]). Roughly speaking, the measure framing on an arbitrary directed graph  $G$  is to attach a Borel measure space  $X = (X, \mathcal{B}_X, \mu)$ , equipped with a topological space  $X$ , a Borel  $\sigma$ -algebra  $\mathcal{B}_X$  of  $X$ , and the (bounded) Borel measure  $\mu$ , on  $G$ .

By understanding the combinatorial object  $G$ , as a discrete topological space  $G = V(G) \cup E(G)$ , we can construct a product topological space  $G_X = X \times G$ , called the *(measure) framed graph of  $G$  framed by  $X$* . In particular,  $X$  is said to be the *frame of  $G_X$* . Then the set

$$\mathbb{G}_X = \mathcal{B}_X \times \mathbb{G},$$

forms a (categorical) groupoid in the sense of [38] (also, see Subsection 2.6 below), under the binary operation,

$$(B_1, w_1)(B_2, w_2) = (B_1 \cap B_2, w_1 w_2),$$

for all  $B_1, B_2 \in \mathcal{B}_X$ , and  $w_1, w_2 \in \mathbb{G}$ . The groupoidal property (or the admissibility) of the graph groupoid  $\mathbb{G}$  governs that (resp., framed admissibility) of  $\mathbb{G}_X$ . Then, the pair  $\mathbb{G}_X = (\mathbb{G}_X, \cdot)$  is a well-defined groupoid and we call it the *framed (graph) groupoid of  $G$ , with the frame  $X$* . This construction, itself, is interesting, since it provides a way to construct groupoids, with uncountably many elements. For instance, if we take a frame  $X$  by  $([0, 1], \mathcal{B}_{[0,1]}, \mu)$ , where  $[0, 1]$  is the closed interval in  $\mathbb{R}$ , and  $\mathcal{B}_{[0,1]}$  is the Borel  $\sigma$ -algebra, generated by all closed subsets of  $[0, 1]$ , and  $\mu$  is the usual Lebesgue measure on  $\mathcal{B}_{[0,1]}$ , then the framed groupoid  $\mathbb{G}_X$ , for any graph groupoid  $\mathbb{G}$ , contains uncountably infinitely many elements.

The main result of [42] is the characterization of the groupoid  $W^*$ -algebras, generated by framed groupoids: If  $M_{G_X} = \overline{\mathbb{C}[\mathbb{G}_X]}^w$  is the groupoid von Neumann algebra, generated by a framed groupoid  $\mathbb{G}_X$ , then

$$M_{G_X} \stackrel{* \text{-isomorphic}}{=} M_X \otimes_{\mathbb{C}} M_G,$$

where

$$M_X = L^\infty(X) = L^\infty(X, \mu),$$

and

$$M_G = \overline{\mathbb{C}[G]}^w$$

is the usual graph von Neumann algebra of  $G$ , in the sense of [3] through [17]. Here, we construct *framed fractaloids*, which is fractaloids, induced by the measure framing.

## 2. Definitions

In this section, we provide the motivation of this paper. Measure (space) framing on graphs and the study of fractals on graphs have been studied independently. In this paper, we provide a connection between them.

**2.1. Measure framing on graphs.** Let  $A$  be a  $C^*$ - (or  $W^*$ -) algebra in a ring  $B(H)$  of (bounded linear) operators on a Hilbert space  $H$ . Let  $\Gamma$  be a group and assume that there exists a *group action*  $\gamma$  of  $\Gamma$ , acting on  $A$ , in the sense that: (i)  $\gamma_g$  are  $*$ -automorphisms of  $A$ , for all  $g \in \Gamma$ , and (ii) for any  $g_1, g_2 \in \Gamma$ ,

$$\gamma_{g_1} \circ \gamma_{g_2} = \gamma_{g_1 g_2},$$

where  $(\circ)$  is the usual composition. Then, the triple  $(A, \Gamma, \gamma)$  is called a *group  $C^*$ - (resp.,  $W^*$ -) dynamical system*. It is well-known that such a group dynamical system  $(A, \Gamma, \gamma)$  generates the corresponding *group  $C^*$ - (resp.,  $W^*$ -) crossed product algebra*  $\mathcal{A} = A \times_\gamma \Gamma$ , and these group crossed product operator algebraic structures have been widely studied.

Let  $\mathbb{R}$  be the time axis equipped with the binary operation, the usual addition  $(+)$ , i.e., we have a group  $\mathbb{R} = (\mathbb{R}, +)$ . Assume that there exists a group action  $\gamma$  of  $\mathbb{R}$  acting on  $A$ , such that

- (i)  $\gamma_t : A \rightarrow A$  are  $*$ -automorphisms for all  $t \in \mathbb{R}$ ,
- (ii)  $\gamma_0(a) = a$ , for all  $a \in A$ ,

- (iii)  $\gamma_t(1_A) = 1_A$ , for all  $t \in \mathbb{R}$ ,
- (iv)  $\gamma_{t_1} \circ \gamma_{t_2} = \gamma_{t_1+t_2}$ , for all  $t_1, t_2 \in \mathbb{R}$ .

Then, in particular, the triple  $(A, \mathbb{R}, \gamma)$  is said to be a *dynamical system with flow (flowed by  $\mathbb{R}$ )*.

More generally, let  $A$  be given as above, and let  $\mathcal{X}$  be a (categorical) *groupoid*. Assume that, there exists a *groupoid action*  $\alpha$  of  $\mathcal{X}$  satisfying that: (i)  $\alpha_x$  are  $*$ -endomorphisms on  $A$ , for all  $x \in \mathcal{X}$ , and (ii) for any  $x_1, x_2 \in \mathcal{X}$ ,

$$\alpha_{x_1} \circ \alpha_{x_2} = \alpha_{x_1 x_2}.$$

Then similar to the group case, we have a *groupoid  $C^*$ - (resp.,  $W^*$ -) dynamical system*  $(A, \mathcal{X}, \alpha)$ , and the corresponding *groupoid crossed product algebra*  $\mathbb{A} = A \times_{\alpha} \mathcal{X}$ . Since all groups are groupoids, groupoidal version of dynamical systems, and crossed product algebras are the enlarged concepts for group version of them.

In particular, if  $A$  is given as before, and  $\mathbb{G}$  is a “graph groupoid” (in the sense of Subsection 3.2 below), then we call the dynamical system  $(A, \mathbb{G}, \alpha)$ , a *graph dynamical system*.

Let  $X = (X, \mathcal{B}_X, \mu)$  be an arbitrary given Borel measure space, and let  $(A, \mathcal{X}, \alpha)$  be a groupoid dynamical system. Then, the calculus on  $X$  may / can be affected (or changed also) by the dynamics of  $A$ . To study such an affection (or a change), we introduce the “framing” on graphs in Section 4 below (also, see [20]). Roughly speaking, the calculus on  $X$  is framed on  $\mathcal{X}$ . Then, we can consider the calculus on  $\mathcal{X}$ , preserving the calculus on  $X$ .

In “our” representations in the sense of Section 4, the groupoid algebras generated by framed groupoids are characterized as tensor products. This means that the changes of the calculus on  $X$  are understood to be tensorized under certain representations for framed groupoids.

**2.2. Fractals on graphs.** While the measure framing on graphs have been studied, the *fractal property*, which is called the *fractality*, for short, of graphs have also been studied (see [9], [12], [15], [16], [19], and [21]). In [13] and [14], we showed that, if there is a certain kind of family  $\mathcal{G}$  of partial isometries on a Hilbert space  $H$ , then  $\mathcal{G}$  induces a directed graph, called the *corresponding graph  $G$  of  $\mathcal{G}$* , and it generates a groupoid  $\mathbb{G}_{\mathcal{G}}$  embedded in  $B(H)$ . It is easy to check that the graph groupoid  $\mathbb{G}$  of  $G$  is groupoid-isomorphic to  $\mathbb{G}_{\mathcal{G}}$ , i.e., the operator-algebraic structures of  $\mathcal{G}$ , embedded in  $B(H)$ , is explained by the elements of  $\mathbb{G}$ . However, the analysis of  $\mathbb{G}_{\mathcal{G}}$  is still very complicated. So, the relatively easy (but not so easy) cases were needed. The first example constructed under this idea was the groupoids  $\mathbb{G}_{\mathcal{G}}$  with fractality.

So, independently, graph groupoids with fractality, called the *graph fractaloids*, have been studied. We call the graphs generating graph fractaloids, the *fractal graphs*. We defined the operator  $T_G$ , induced by a fractal graph  $G$ , in a certain von Neumann algebra  $M_G$ , and observed the free distributional data of  $T_G$ , by computing the free moments of it. Such data give us the spectral information of  $T_G$ , and hence explain how the graph fractaloid  $\mathbb{G}$  of  $G$  works in the von Neumann algebra  $M_G$ .

In particular, in [9], we showed that, there are sufficiently many fractal graphs generating graph fractaloids, which are not just fractal groups (in the sense of [39]).

**2.3. Frames and fractals.** In this paper, as application of the study of framed graphs and framed groupoids, we introduce graph fractaloids and the corresponding “framed” fractaloids. It demonstrates nicely how our measure framing works in an operator algebraic structure. In particular, we can check how the fractality of graph fractaloids preserves the frames, which are measure spaces, through fractal graphs, in  $B(H)$ .



### 3. Background

In this section, we introduce the concepts we will use in context.

**3.1. Free probability.** Let  $B \subset A$  be von Neumann algebras with  $1_B = 1_A$ , and assume that there is a *conditional expectation*  $E_B : A \rightarrow B$ , satisfying that: (i)  $E_B$  is a  $(\mathbb{C}-)$ linear map, (ii)  $E_B(b) = b$ , for all  $b \in B$ , (iii)  $E_B(b_1 a b_2) = b_1 E_B(a) b_2$ , for all  $b_1, b_2 \in B$  and  $a \in A$ , (iv)  $E_B$  is continuous under the given topologies of  $A$  and  $B$ , and (v)  $E_B(a^*) = E_B(a)^*$  in  $B$ , for all  $a \in A$ .

The algebraic pair  $(A, E_B)$  is said to be a *B-valued  $W^*$ -probability space*. Every operator in  $(A, E_B)$  is called a *B-valued (free) random variable*. Any *B-valued random variables* have their *B-valued free distributional data*: *B-valued \*-moments* and *B-valued \*-cumulants* of them. Suppose  $a_1, \dots, a_s$  are *B-valued random variables* in  $(A, E_B)$ , where  $s \in \mathbb{N}$ . The  $(i_1, \dots, i_n)$ -th *joint B-valued \*-moments* of  $a_1, \dots, a_s$  are defined by

$$E_B((b_1 a_{i_1}^{r_{i_1}})(b_2 a_{i_2}^{r_{i_2}}) \dots (b_n a_{i_n}^{r_{i_n}})),$$

and the  $(j_1, \dots, j_k)$ -th *joint B-valued \*-cumulants* of  $a_1, \dots, a_s$  are defined by

$$k_k^B \left( (b_1 a_{j_1}^{r_{j_1}}), \dots, (b_k a_{j_k}^{r_{j_k}}) \right) = \sum_{\pi \in NC(k)}^{(N)} E_{B:\pi} \left( b_1 a_{j_1}^{r_{j_1}}, \dots, b_k a_{j_k}^{r_{j_k}} \right) \mu(\pi, 1_k),$$

for all  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$  and for all  $(j_1, \dots, j_k) \in \{1, \dots, s\}^k$ , for  $n, k \in \mathbb{N}$ , where  $b_j \in B$  are arbitrary and  $r_{i_1}, \dots, r_{i_n}, r_{j_1}, \dots, r_{j_k} \in \{1, *\}$ , and  $NC(k)$  is the *lattice of all noncrossing partitions* with its minimal element  $0_k = \{(1), (2), \dots, (k)\}$  and its maximal element  $1_k = \{(1, 2, \dots, k)\}$ , for all  $k \in \mathbb{N}$ , and  $\mu$  is the Moebius functional in the incidence algebra  $\mathcal{I}$ . Here,  $E_{B:\pi}(\dots)$  is the *partition-depending B-valued moment*.

For example, if  $\pi = \{(1, 4), (2, 3), (5)\}$  in  $NC(5)$ , then

$$E_{B:\pi}(a_1, a_2, a_3, a_4, a_5) = E_B(a_1 E_B(a_2 a_3) a_4) E_B(a_5).$$

Recall that the lattice  $NC(n)$  of all noncrossing partitions over  $\{1, \dots, n\}$  has its partial ordering “ $\leq$ ,”

$$\pi \leq \theta \stackrel{\text{def}}{\Leftrightarrow} \forall \text{ a block } V \text{ in } \pi, \exists \text{ a block } B \text{ in } \theta \text{ s.t., } V \subseteq B,$$

for  $\pi, \theta \in NC(n)$ , where “ $\subseteq$ ” means the usual set inclusion, for all  $n \in \mathbb{N}$ . Also, recall that the *incidence algebra*  $\mathcal{I}$  is the collection of all functionals

$$\xi : \bigcup_{n=1}^{\infty} (NC(n) \times NC(n)) \rightarrow \mathbb{C}$$

satisfying that  $\xi(\pi, \theta) = 0$ , whenever  $\pi > \theta$ , with the usual function addition (+) and the convolution (\*) defined by

$$(\xi_1 * \xi_2)(\pi, \theta) \stackrel{\text{def}}{=} \sum_{\pi \leq \sigma \leq \theta}^{(N)} \xi_1(\pi, \sigma) \xi_2(\sigma, \theta),$$

for all  $\xi_1, \xi_2 \in \mathcal{I}$ . If we define the *zeta functional*  $\zeta \in \mathcal{I}$  by

$$\zeta(\pi, \theta) = 1, \text{ for all } \pi \leq \theta \text{ in } NC(n), \text{ for all } n \in \mathbb{N},$$

then its convolution-inverse in  $\mathcal{I}$  is the *Moebius functional*  $\mu$ . Thus, the Moebius functional  $\mu$  satisfies that

$$\mu(0_n, 1_n) = (-1)^{n-1} c_{n-1} \text{ and } \sum_{\pi \in NC(n)}^{(N)} \mu(\pi, 1_n) = 0,$$

where  $c_k \stackrel{\text{def}}{=} \frac{1}{k+1} \binom{2k}{k}$  is the *k-th Catalan number*, for all  $k \in \mathbb{N}$ .

The *B*-valued freeness on  $(A, E_B)$  is characterized by the *B*-valued \*-cumulants. Let  $A_1$  and  $A_2$  be  $W^*$ -subalgebras of  $A$  having their common  $W^*$ -subalgebra  $B$ . We say that  $A_1$  and  $A_2$  are free over  $B$  in

$(A, E_B)$ , if all mixed  $B$ -valued  $*$ -cumulants of  $A_1$  and  $A_2$  vanish. The subsets  $X_1$  and  $X_2$  of  $A$  are said to be free over  $B$  in  $(A, E_B)$ , if the  $W^*$ -subalgebras  $vN(X_1, B)$  and  $vN(X_2, B)$  of  $A$  are free over  $B$  in  $(A, E_B)$ , where  $vN(S_1, S_2)$  means the von Neumann algebra generated by sets  $S_1$  and  $S_2$ . Similarly, we say that the  $B$ -valued random variables  $x$  and  $y$  are free over  $B$  in  $(A, E_B)$ , if the subsets  $\{x\}$  and  $\{y\}$  are free over  $B$  in  $(A, E_B)$ .

Let  $A_1$  and  $A_2$  be  $W^*$ -subalgebra of  $A$  containing their common  $W^*$ -subalgebra  $B$ , and assume that they are free over  $B$  in  $(A, E_B)$ . Then, we can construct a  $W^*$ - algebra  $vN(A_1, A_2)$  of  $A$  generated by  $A_1$  and  $A_2$ . We denote it by  $A_1 *_B A_2$ . Suppose there exists a family  $\{A_i : i \in \Lambda\}$  of  $W^*$ -subalgebras of  $A$  containing their common  $W^*$ -subalgebra  $B$ . If they generate  $A$  and if they are free over  $B$  from each other in  $(A, E_B)$ , then the von Neumann algebra  $A$  is the  $B$ -valued free product algebra  $*_{i \in \Lambda} A_i$ .

Suppose a von Neumann algebra  $A$  is a  $B$ -free product algebra  $*_{i \in \Lambda} A_i$ .

Then  $A$  is Banach-space isomorphic to the Banach space

$$B \oplus \left( \bigoplus_{n=1}^{\infty} \left( \bigoplus_{i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n} (A_{i_1}^o \otimes_B \dots \otimes_B A_{i_n}^o) \right) \right)$$

with

$$A_{i_j}^o \stackrel{\text{def}}{=} A_{i_j} \ominus B, \text{ for all } j = 1, \dots, n,$$

where  $\otimes_B$  is the  $B$ -valued tensor product.

Let  $a_k$  be “self-adjoint”  $B$ -valued random variables in a  $B$ -valued  $W^*$ -probability space  $(A_k, E_B^k)$ , for  $k = 1, 2$ . We say that the two self-adjoint operators  $a_1$  and  $a_2$  are *identically distributed over  $B$* , if

$$E_B^1(a_1^k) = E_B^2(a_2^k) \text{ in } B, \text{ for all } k \in \mathbb{N}.$$

**3.2. Graph groupoids.** Let  $G$  be a directed graph with its vertex set  $V(G)$  and its edge set  $E(G)$ . Let  $e \in E(G)$  be an edge connecting a vertex  $v_1$  to a vertex  $v_2$ . Then, we write  $e = v_1 e v_2$ , for emphasizing the initial vertex  $v_1$  of  $e$  and the terminal vertex  $v_2$  of  $e$ . For a graph  $G$ , we can define the oppositely directed graph  $G^{-1}$ , with  $V(G^{-1}) = V(G)$  and  $E(G^{-1}) = \{e^{-1} : e \in E(G)\}$ , where each element  $e^{-1}$  satisfies that  $e = v_1 e v_2$  in  $E(G)$ , with  $v_1, v_2 \in V(G)$ , if and only if  $e^{-1} = v_2 e^{-1} v_1$ , in  $E(G^{-1})$ . This opposite directed edge  $e^{-1} \in E(G^{-1})$  of  $e \in E(G)$  is called the *shadow of  $e$* . Also, this new graph  $G^{-1}$ , induced by  $G$ , is said to be the *shadow of  $G$* . It is clear that  $(G^{-1})^{-1} = G$ .

Define the *shadowed graph*  $\widehat{G}$  of  $G$  by a directed graph with its vertex set  $V(\widehat{G}) = V(G) = V(G^{-1})$  and its edge set  $E(\widehat{G}) = E(G) \cup E(G^{-1})$ , where  $G^{-1}$  is the *shadow of  $G$* . We say that two edges  $e_1 = v_1 e_1 v'_1$  and  $e_2 = v_2 e_2 v'_2$  are *admissible*, if  $v'_1 = v_2$ , equivalently, the finite path  $e_1 e_2$  is well-defined on  $\widehat{G}$ . Similarly, if  $w_1$  and  $w_2$  are finite paths on  $G$ , then we say  $w_1$  and  $w_2$  are *admissible*, if  $w_1 w_2$  is a well-defined finite path on  $G$ , too. Similar to the edge case, if a finite path  $w$  has its initial vertex  $v$  and its terminal vertex  $v'$ , then we write  $w = v_1 w v_2$ . Notice that every admissible finite path is a word in  $E(\widehat{G})$ . Denote the set of all finite path by  $FP(\widehat{G})$ . Then  $FP(\widehat{G})$  is the subset of  $E(\widehat{G})^*$ , consisting of all words in  $E(\widehat{G})$ .

We can construct the *free semigroupoid*  $\mathbb{F}^+(\widehat{G})$  of the shadowed graph  $\widehat{G}$ , as the union of all vertices in  $V(\widehat{G}) = V(G) = V(G^{-1})$  and admissible words in  $FP(\widehat{G})$ , with its binary operation, the *admissibility*.

Naturally, we assume that  $\mathbb{F}^+(\widehat{G})$  contains the empty word  $\emptyset$ . Remark that some free semigroupoid  $\mathbb{F}^+(\widehat{G})$  of  $\widehat{G}$  does not contain the empty word; for instance, if a graph  $G$  is a one-vertex-multi-edge graph, then the shadowed graph  $\widehat{G}$  of  $G$  is also a one-vertex-multi-edge graph, and it induces the free semigroupoid  $\mathbb{F}^+(\widehat{G})$ , which does not have the empty word. However, in general, if  $|V(G)| > 1$ , then  $\mathbb{F}^+(\widehat{G})$  always contain the empty word. Thus, if there is no confusion, we always assume the empty word  $\emptyset$  is contained in the free semigroupoid  $\mathbb{F}^+(\widehat{G})$  of  $\widehat{G}$ .

By defining the *reduction* (RR) on  $\mathbb{F}^+(\widehat{G})$ , we can construct the graph groupoid  $\mathbb{G}$ .

**Definition 3.1.** Let  $G$  be a countable directed graph with its shadowed graph  $\widehat{G}$ . The graph groupoid  $\mathbb{G}$  is a set of all “reduced” words in the edge set  $E(\widehat{G})$  of  $\widehat{G}$ , equipped with the inherited admissibility on the free semigroupoid  $\mathbb{F}^+(\widehat{G})$ , where the reduction (RR) on  $\mathbb{G}$  is defined as follows:

$$(RR) \quad ww^{-1} = v \quad \text{and} \quad w^{-1}w = v',$$

for all  $w = v w' v' \in \mathbb{G}$ , with  $v, v' \in V(\widehat{G})$ .

In fact, graph groupoids are indeed categorial groupoids.

**3.3. Graph von Neumann algebras.** In this section, we briefly introduce graph von Neumann algebras of the graphs (see [3] through [2]). Let  $G$  be a directed graph with its graph groupoid  $\mathbb{G}$ . Then, we can construct the corresponding Hilbert space  $H_G$  by

$$H_G \stackrel{\text{def}}{=} \left( \bigoplus_{v \in V(\widehat{G})} \mathbb{C} \xi_v \right) \oplus \left( \bigoplus_{w \in FP_r(\widehat{G})} \mathbb{C} \xi_w \right) = l^2(\mathbb{G}).$$

Then, this Hilbert space  $H_G$  is called the *graph Hilbert space* induced by  $G$ . Notice that, by definition, the Hilbert space  $H_G$  has its Hilbert basis  $\{\xi_w : w \in FP_r(\widehat{G})\}$ .

Define the *canonical (left) groupoid action*

$$L : \mathbb{G} \rightarrow B(H_G),$$

of  $\mathbb{G}$ , acting on  $H_G$ , by

$$L(w) \stackrel{\text{def}}{=} L_w \in B(H_G), \text{ for all } w \in \mathbb{G},$$

where  $L_w$ 's are the multiplication operators with their symbols  $\xi_w$ , with their adjoint  $L_w^* = L_{w^{-1}}$ , for all  $w \in \mathbb{G}$ .

The pair  $(H_G, L)$  is called the *canonical representation* of  $\mathbb{G}$ .

**Definition 3.2.** Let  $G$  be a graph with its graph groupoid  $\mathbb{G}$ , and let  $(H_G, L)$  be the canonical representation of  $\mathbb{G}$ . Define the groupoid  $W^*$ -algebra  $M_G$  by  $\overline{\mathbb{C}[L(\mathbb{G})]}^w$  in  $B(H_G)$ . This groupoid  $W^*$ -algebra  $M_G$  is called the *graph von Neumann algebra* of  $G$ . Define a  $W^*$ -subalgebra  $D_G$  of  $M_G$  by

$$D_G \stackrel{\text{def}}{=} \bigoplus_{v \in V(\widehat{G})} (\mathbb{C} \cdot R_v).$$

It is called the *diagonal subalgebra* of  $M_G$ .

**Remark 3.1.** In [6], [14], and [15], we observed the right multiplication operators  $R_w$ 's, for all  $w \in \mathbb{G}$ , instead of using left multiplication operators  $L_w$ 's. Then, we can define the right graph von Neumann algebra  $M_G^{op} = \overline{\mathbb{C}[R(\mathbb{G})]}^w$  in  $B(H_G)$ , where  $R : \mathbb{G} \rightarrow B(H_G)$  is the right groupoid action of  $\mathbb{G}$ , acting on  $H_G$ , i.e.,  $R_w \xi_{w'} \stackrel{\text{def}}{=} \xi_{w'w}$ , for

all  $w, w' \in \mathbb{G}$ . Notice that  $M_G^{op}$  and  $M_G$  are anti- $*$ -isomorphic, because indeed, the right graph von Neumann algebra  $M_G^{op}$  is the opposite  $*$ -algebra of the graph von Neumann algebra  $M_G$ . Thus, they share the fundamental properties.

Notice that, every element  $x$  in the graph von Neumann algebra  $M_G$  of  $G$  has its expression,

$$x = \sum_{w \in \mathbb{G}} t_w L_w, \text{ with } t_w \in \mathbb{C}.$$

Let  $D_G$  be the diagonal subalgebra of  $M_G$ . Define the *canonical conditional expectation*

$$E : M_G \rightarrow D_G$$

by

$$E \left( \sum_{w \in \mathbb{G}} t_w L_w \right) \stackrel{\text{def}}{=} \sum_{v \in V(\hat{G})} t_v L_v,$$

for all  $\sum_{w \in \mathbb{G}} t_w L_w \in M_G$ . Then, the pair  $(M_G, E)$  is a  $D_G$ -valued  $W^*$ -probability space over  $D_G$ , in the sense of Voiculescu (see [43] and [51]).

**Definition 3.3.** The  $D_G$ -valued  $W^*$ -probability space  $(M_G, E)$  is called the *graph  $W^*$ -probability space* induced by the given graph  $G$ .

By [3] and [4], we have the following two results:

**Theorem A** (see [3] and [4]). *Let  $M_G$  be the right graph von Neumann algebra of  $G$ . Then, it is  $*$ -isomorphic to the  $D_G$ -valued reduced free product algebra  $*_{D_G}^r M_e$  of the  $D_G$ -free blocks  $M_e$ , where  $e \in E(G)$*

$M_e \stackrel{\text{def}}{=} vN(\mathbb{G}_e, D_G)$  in  $B(H_G)$ , where  $\mathbb{G}_e$  are the subgroupoid of  $\mathbb{G}$ , induced by  $\{e, e^{-1}\}$ , for all  $e \in E(G)$ . □

**Theorem B** (see [4]). Let  $M_G$  be the right graph von Neumann algebra of  $G$ , and let  $\ast_{D_G}^r M_e$  be the  $D_G$ -valued reduced free product algebra of  $M_e$ 's, which is  $\ast$ -isomorphic to  $M_G$ , in  $B(H_G)$ .

(1) If  $e$  is a loop edge, then the corresponding  $D_G$ -free block  $M_e$  is  $\ast$ -isomorphic to the group von Neumann algebra  $L(\mathbb{Z})$ , generated by the infinite cyclic abelian group  $\mathbb{Z}$ , which is also  $\ast$ -isomorphic to the  $L^\infty$ -algebra  $L^\infty(\mathbb{T})$ , where  $\mathbb{T}$  is the unit circle in  $\mathbb{C}$ .

(2) If  $e$  is a non-loop edge, then  $M_e$  is  $\ast$ -isomorphic to the matricial algebra  $M_2(\mathbb{C})$ , consisting of all  $(2 \times 2)$ -matrices. □

**3.4. Radial operators of graph groupoids.** Let  $G$  be a countable directed graph with its graph groupoid  $\mathbb{G}$ , and let  $M_G$  be the graph von Neumann algebra of  $G$ . Define an operator  $T_G$  by

$$T_G \stackrel{\text{def}}{=} \sum_{e \in E(\widehat{G})} L_e = \sum_{e \in E(G)} (L_e + L_{e^{-1}}),$$

in  $M_G$ .

**Definition 3.4.** This operator  $T_G$  is called the *radial operator* of  $\mathbb{G}$ .

Let  $G$  be a countable directed graph. Then, every vertex  $v$  of  $G$  has the following quantities:

$$\text{deg}_{out}(v) \stackrel{\text{def}}{=} |\{e \in E(G) : e = ve\}|,$$

called the *out-degree* of  $v$  (in  $G$ )

$$\text{deg}_{in}(v) \stackrel{\text{def}}{=} |\{e \in E(G) : e = ev\}|,$$

called the *in-degree* of  $v$  (in  $G$ ), and



$$\text{deg}(v) \stackrel{\text{def}}{=} \text{deg}_{out}(v) + \text{deg}_{in}(v),$$

called the *degree* of  $v$  (in  $G$ ), for all  $v \in V(G)$ .

Now, let  $G$  be a connected, locally finite graph. Recall that a graph  $G$  is *connected*, if for any pair  $(v, v')$  of “distinct” vertices, there always exists at least one element  $w$  in the graph groupoid  $\mathbb{G}$  of  $G$ , such that  $w = v w v'$ , and  $w^{-1} = v' w^{-1} v$ . Recall also that a graph  $G$  is *locally finite*, there are only finitely many incident edges for every vertex of  $G$ , equivalently,  $G$  is locally finite, if and only if  $\text{deg}(v) < \infty$ , for all  $v \in V(G)$ .

For the given connected locally finite graph  $G$ , define the quantity  $N$  by

$$N \stackrel{\text{def}}{=} \max\{\text{deg}_{out}(v) : v \in V(G)\} \text{ in } G.$$

Then, by the locally finiteness of  $G$ , this number  $N$  is less than  $\infty$ . In [15], we computed the  $D_G$ -valued moments  $E(T_G^n)$  and the  $D_G$ -valued cumulants  $k_n(T_G, \dots, T_G)$ , for all  $n \in \mathbb{N}$ . Also, in [14], we computed them, where  $\mathbb{G}$  is a graph fractaloid.

By the very definition, the radial operators are self-adjoint in  $M_G$ , in the sense that  $T_G^* = T_G$ . So, the  $D_G$ -valued moments and the  $D_G$ -valued cumulants of  $T_G$  contain full free distributional data of  $T_G$ . Such data for  $T_G$  show how  $\mathbb{G}$  works on  $H_G$ .

#### 4. Measure Framing on Graphs

Throughout this section, let  $G$  be an arbitrary directed graph. However, for our main purpose, the readers may / can assume all graphs are connected and locally finite. However, we emphasize that the following results of this section is applicable for the general cases. Let  $X = (X, \mathcal{B}_X, \mu)$  be a Borel space, where  $X$  is a topological space,  $\mathcal{B}_X$  is a Borel  $\sigma$ -algebra of  $X$ , and  $\mu$  is a Borel measure on  $\mathcal{B}_X$ .

**4.1. Framed graphs and framed groupoids.** From now on, regard the combinatorial object  $G$  as a discrete topological space  $V(G) \cup E(G)$ , also denoted by  $G$ . Then, we can construct the topological space

$$G_X = X \times G,$$

under the product topology of  $X$  and  $G$ , i.e., the topological space  $G_X$  is the set

$$X \times G = \{(x, y) : x \in X, y \in V(G) \cup E(G)\},$$

equipped with the product topology of  $X$  and  $G = V(G) \cup E(G)$ .

**Definition 4.1.** The topological space  $G_X$  is called the *framed graph* of  $G$  with the frame  $X$ .

The elements  $(x, y)$  of a framed graph  $G_X$  can be understand as the elements  $x$  of  $X$  having their movements  $y$  determined by (the direction of)  $G$ . If  $y \in G$  is a vertex, then  $(x, y) \in G_X$  can be regarded as  $x \in X$  at the position  $y$  (without movement). So, the elements  $(x, y_1)$  and  $(x, y_2)$  are distinct elements in  $G_X$ . If  $y \in G$  is an edge  $y = v y v'$ , with  $v, v' \in V(G)$ , then  $(x, y)$  is  $x \in X$  moving from the position  $v$  to the position  $v'$ .

Also, the elements  $(x, y)$  of  $G_X$  can be understand the movements  $y \in G$ , with their properties represented by  $x \in X$ . For our purpose, we consider the elements  $(x, y)$  of  $G_X$  as the movements  $y \in G$  with their properties  $x \in X$ . So, similarly, the elements  $(x_1, y)$  and  $(x_2, y)$  of  $G_X$  are regarded as distinct elements: Even though, they have same movements determined by  $y \in G$ , their properties are distinct, whenever  $x_1 \neq x_2$  in  $X$ .

Then define the Cartesian product

$$\mathbb{G}_X = \mathcal{B}_X \times \mathbb{G},$$

induced by  $G_X$ . As a set,  $\mathbb{G}_X$  is

$$\{(B, w) : B \in \mathcal{B}_X, w \in \mathbb{G}\}.$$

The elements  $(B, w)$  of  $\mathbb{G}_X$  are regarded as the movements determined by the movements  $e_1, \dots, e_k \in E(\widehat{G})$ , where  $w \in \mathbb{G}$  is the “reduced” word  $e_1 \dots e_k$  in  $\mathbb{F}^+(\widehat{G})$ , for  $k \in \mathbb{N}$ , preserving or maintaining the property  $B \in \mathcal{B}_X$ , represented by the measure  $\mu(B)$ . Or, alternatively, we may / can understand  $(B, w) \in \mathbb{G}_X$  as the property  $B \in \mathcal{B}_X$ , represented by  $\mu(B)$ , preserved (or maintained) by the movement  $w = e_1 \dots e_k \in \mathbb{G}$ .

Now, define the binary operation  $(\cdot)$  on  $\mathbb{G}_X$  by

$$(B_1, w_1)(B_2, w_2) = (B_1 \cap B_2, w_1w_2),$$

for all  $(B_k, w_k) \in \mathbb{G}_X$ , for  $k = 1, 2$ , with the *empty element*  $\emptyset_X$  of  $\mathbb{G}_X$

$$(\emptyset, w) = \emptyset_X = (B, \emptyset),$$

for all  $w \in \mathbb{G}$ , and  $B \in \mathcal{B}_X$ , where  $\emptyset$  is the empty set, and  $\emptyset$  is the empty element of  $\mathbb{G}$ . Such a binary operation is called the *framed admissibility* on the set  $\mathbb{G}_X$ .

Notice that the pair  $\mathbb{G}_X = (\mathbb{G}_X, \cdot)$  of the set  $\mathbb{G}_X$  and the framed admissibility  $(\cdot)$  is indeed a groupoid with the groupoid inverses;

$$(B, w)^{-1} \stackrel{\text{def}}{=} (B, w^{-1}),$$

for all  $(B, w) \in \mathbb{G}_X$ , where  $w^{-1} \in \mathbb{G}$  is the shadow of  $w \in \mathbb{G}$ . We call the groupoid inverse  $(B, w)^{-1}$  of  $(B, w)$ , the *framed shadow* of  $(B, w)$ .

Define now subsets  $V(\widehat{G}_X)$  and  $FP_r(\widehat{G}_X)$  of  $\mathbb{G}_X$  by

$$V(\widehat{G}_X) \stackrel{\text{def}}{=} \{(B, v) : B \in \mathcal{B}_X, v \in V(\widehat{G})\},$$

and

$$FP_r(\widehat{G}_X) \stackrel{\text{def}}{=} \{(B, w) : B \in \mathcal{B}_X, w \in FP_r(\widehat{G})\},$$

where  $\widehat{G}$  is the shadowed graph of  $G$ , in the sense of Subsection 3.2. We call  $V(\widehat{G}_X)$ ,  $FP_r(\widehat{G}_X)$ , the *framed vertex set* and the *framed reduced finite path set*, respectively. Clearly, without loss of generality, the notation  $\widehat{G}_X$  can be understood as the framed graph  $X \times \widehat{G}$ . This framed graph  $\widehat{G}_X = X \times \widehat{G}$  is called the *framed shadowed graph* of the framed graph  $G_X$ . Similarly, if we use the notation  $G_X^{-1}$ , it means the framed graph  $X \times G^{-1}$ , where  $G^{-1}$  is the shadow of  $G$ . We call  $G_X^{-1}$ , the *framed shadow* of  $G_X$ . (Notice here that the framed structures  $G_X$ ,  $\widehat{G}_X$ , and  $G_X^{-1}$  are not purely combinatorial objects. They are topological spaces, determined by  $X$  and  $G$ , or  $\widehat{G}$ , or  $G^{-1}$ , respectively.)

**Definition 4.2.** The pair  $(\mathbb{G}_X, \cdot)$  of the set  $\mathbb{G}_X$  and the framed admissibility  $(\cdot)$  is called the *framed (graph) groupoid* of  $G$  with the frame  $X$ . We denote this pair simply by  $\mathbb{G}_X$ .

**Remark 4.1.** (1) All elements of the framed groupoid  $\mathbb{G}_X$  are the algebraic objects equipped with certain property represented by the Borel sets, as we discussed before. So, the framed admissibility on  $\mathbb{G}_X$  means that, whenever the algebraic object  $w$  with the property  $B$  (i.e.,  $(B, w) \in \mathbb{G}_X$ ) meets with the admissible successor  $y$  with the property  $C$  (i.e.,  $(C, y) \in \mathbb{G}_X$ ), then  $w$  follows  $y$ , determined by  $wy \in \mathbb{G}$ , with the restricted property  $B \cap C$ .

(2) The study of framed graphs and framed groupoids are interesting, since it provides a way to create groupoids with “uncountably” infinitely many elements. For instance, if we take a measure space  $X$  as a standard Borel measure space  $((0, 1), \mathcal{B}, \rho)$ , where  $\rho$  is the usual Lebesgue measure on the open interval  $(0, 1)$ , then for any countable directed graphs  $G$  with their graph groupoids  $\mathbb{G}$ , the framed groupoids  $\mathbb{G}_X$  contain uncountably infinitely many elements, as groupoids.

By the very definition, we have the following proposition:

**Proposition 4.1.** *Let  $G^{(k)}$  be graphs with their graph groupoids  $\mathbb{G}^{(k)}$ , and let  $X_k$  be measure spaces, for  $k = 1, 2$ . Let  $G_{X_k}^{(k)}$  be framed graphs with their framed groupoids  $\mathbb{G}_{X_k}^{(k)}$ , for  $k = 1, 2$ . If  $X_k$  are equivalent, and if the shadowed graphs  $\widehat{G^{(k)}}$  of  $G^{(k)}$  are graph-isomorphic, then the framed groupoids  $\mathbb{G}_{X_k}^{(k)}$  are groupoid-isomorphic.  $\square$*

**4.2. Canonical representation of framed groupoids.** Similar to the canonical representation of graph groupoids, we consider that of “framed” groupoids. Throughout this section, let  $G$  be a countable directed graph with its graph groupoid  $\mathbb{G}$ , and  $X = (X, \mathcal{B}_X, \mu)$ , a fixed Borel measure space. Recall that, the graph  $G$  induces its graph Hilbert space

$$H_G = l^2(\mathbb{G}),$$

and the Borel space  $X$  induces its corresponding Hilbert space

$$H_X = L^2(X, \mu),$$

consisting of all square integrable measurable functions on  $X$ . Recall also that  $H_G$  has its inner product,

$$\langle \xi_{w_1}, \xi_{w_2} \rangle_G \stackrel{\text{def}}{=} \delta_{w_1, w_2},$$

for all Hilbert basis elements  $\xi_{w_k}$ , for all  $w_k \in \mathbb{G}$ , for  $k = 1, 2$ , where  $\delta$  means the Kronecker delta, and  $H_X$  has its inner product,

$$\langle g_1, g_2 \rangle_2 \stackrel{\text{def}}{=} \int_X g_1 \overline{g_2} d\mu,$$

for all  $g_k \in H_X$ , for  $k = 1, 2$ , in particular,

$$\langle \chi_{B_1}, \chi_{B_2} \rangle_2 = \int_X \chi_{B_1} \overline{\chi_{B_2}} d\mu = \mu(B_1 \cap B_2),$$

for all  $B_1, B_2 \in \mathcal{B}_X$ , where

$$\chi_B(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } x \in X, \\ 0, & \text{otherwise,} \end{cases}$$

for all  $B \in \mathcal{B}_X$ .

By construction, we can determine the representation of the framed groupoid  $\mathbb{G}_X$  of  $G$ , with the frame  $X$ , canonically. Define the Hilbert space  $H_{G_X}$  by

$$H_{G_X} \stackrel{\text{def}}{=} H_X \otimes H_G.$$

The inner product  $\langle, \rangle$  is defined by

$$\begin{aligned} \langle \xi_{(B_1, w_1)}, \xi_{(B_2, w_2)} \rangle &\stackrel{\text{def}}{=} \mu(B_1 \cap B_2) \delta_{w_1, w_2}, \\ &= \langle \chi_{B_1}, \chi_{B_2} \rangle_2 \langle \xi_{w_1}, \xi_{w_2} \rangle_G, \end{aligned}$$

for all  $(B_k, w_k) \in \mathbb{G}_X$ , for  $k = 1, 2$ . For instance,

$$\langle \xi_{(B, w)}, \xi_{(B, w)} \rangle = \mu(B), \text{ for all } (B, w) \in \mathbb{G}_X,$$

and hence

$$\|\xi_{(B, w)}\| = |\mu(B)|, \text{ for all } (B, w) \in \mathbb{G}_X.$$

**Definition 4.3.** This Hilbert space  $H_{G_X}$  is called the *framed graph Hilbert space of  $G_X$* .

Then all elements  $(B, w) \in \mathbb{G}_X$  act on  $H_{G_X}$ , as the multiplication operators with their symbols  $\xi_{(B, w)}$ , for all  $(B, w) \in \mathbb{G}_X$ , i.e., we can define the groupoid action

$$L : \mathbb{G}_X \rightarrow B(H_{G_X})$$

by

$$L(B, w) \stackrel{\text{def}}{=} L_{(B,w)},$$

the multiplication operator with its symbol  $\xi_{(B,w)} \in H_{G_X}$ . Notice that the operator  $L_{(B,w)}$  acts on  $H_{G_X}$  by

$$\chi_B \otimes L_w \text{ on } H_X \otimes H_G = H_{G_X},$$

where  $\chi_B$  is the characteristic function, for all  $B \in \mathcal{B}_X$ , and  $q \in X$ , and  $L_w$  is the multiplication operator with  $\xi_w$  on  $H_G$ . Then  $L$  is indeed a groupoid action of  $\mathbb{G}_X$ , since:

$$\begin{aligned} L((B_1, w_1)(B_2, w_2)) &= L_{(B_1 \cap B_2, w_1 w_2)} \\ &= L_{(B_1, w_1)} L_{(B_2, w_2)} \\ &= L_{(B_1, w_1)} L_{(B_2, w_2)} \\ &= L_{(B_1, w_1)} L_{(B_2, w_2)}. \end{aligned}$$

This action  $L$  is called the *canonical (left) groupoid action of  $\mathbb{G}_X$* .

**Definition 4.4.** The pair  $(H_{G_X}, L)$ , with the framed graph Hilbert space  $H_{G_X}$ , and canonical groupoid action of  $\mathbb{G}_X$ , is called the *canonical (left) representation of the framed groupoid  $\mathbb{G}_X$* . Define the groupoid von Neumann algebra  $M_{G_X}$  by

$$M_{G_X} \stackrel{\text{def}}{=} \overline{\mathbb{C}[L(\mathbb{G}_X)]}^w,$$

as a  $W^*$ -subalgebra of  $B(H_{G_X})$ . Then, it is called the *framed graph von Neumann algebra of  $G$* , with the frame  $X$  (or, of  $G_X$ ).

Let  $M_{G_X}$  be the framed graph von Neumann algebra of  $G_X$ . Define a  $W^*$ -subalgebra  $\mathcal{M}_{G:X}$  of  $B(H_X) \otimes_{\mathbb{C}} B(H_G)$  by the tensor product algebra,

$$\mathcal{M}_{G:X} \stackrel{\text{def}}{=} M_X \otimes_{\mathbb{C}} M_G$$

of  $M_X = L^\infty(X, \mu)$ , and the usual graph von Neumann algebra  $M_G = \overline{\mathbb{C}[\mathbb{G}]}^w$ .

**Definition 4.5.** Now, define the linear map

$$\Phi : M_{G_X} \rightarrow \mathcal{M}_{G:X}$$

by

$$\Phi \left( \sum_{(B,w) \in \mathbb{G}_X} t_{(B,w)} L_{(B,w)} \right) \stackrel{\text{def}}{=} \sum_{(B,w) \in \mathcal{B}_X \times \mathbb{G}} t_{(B,w)} (\chi_B \otimes L_w),$$

for all  $\sum_{(B,w) \in \mathbb{G}_X} t_{(B,w)} L_{(B,w)} \in M_{G_X}$ , with  $t_{(B,w)} \in \mathbb{C}$ .

Notice that every element  $y$  of  $M_{G_X}$  has its expression

$$y = \sum_{(B,w) \in \mathbb{G}_X} s_{(B,w)} L_{(B,w)}, \text{ with } s_{(B,w)} \in \mathbb{C}.$$

The map  $\Phi$  of the above definition is  $*$ -multiplicative. Thus,  $\Phi$  is a  $*$ -homomorphism. Also, by the very definition of the framed groupoid  $\mathbb{G}_X$ , it is equipotent (or bijective) to the set  $\mathcal{B}_X \times \mathbb{G}$ . So, the map  $\Phi$  is a generator-preserving bijection, and hence  $\Phi$  is a  $*$ -isomorphism. Thus, the von Neumann algebras  $M_{G_X} \subseteq B(H_{G_X})$  and  $\mathcal{M}_{G:X} \subseteq B(H_X \otimes H_G)$  are  $*$ -isomorphic.

**Observation.**  $M_{G_X} \stackrel{*-\text{iso}}{=} M_X \otimes_{\mathbb{C}} M_G$ . □

By defining the *framed diagonal subalgebra*  $D_{G_X}$  of  $M_{G_X}$  by

$$D_{G_X} \stackrel{\text{def}}{=} \bigoplus_{(B,v) \in V(\widehat{G_X})} \mathbb{C} L_{(B,v)} \stackrel{*-\text{iso}}{=} M_X \otimes_{\mathbb{C}} D_G,$$



where  $D_G$  is the diagonal subalgebra of the usual graph von Neumann algebra  $M_G$  of  $G$ , we can define the conditional expectation

$$E_0 : M_{G_X} \rightarrow D_{G_X}$$

by

$$E_0 \left( \sum_{(B,w) \in G_X} t_{(B,w)} L_{(B,w)} \right) \stackrel{\text{def}}{=} \sum_{(B,v) \in V(\widehat{G_X})} t_{(B,v)} L_{(B,v)}.$$

This also shows that there do exist the amalgamated free probability on  $M_{G_X}$ , too.

By definition of the conditional expectation  $E_0$ , over the framed diagonal subalgebra  $D_{G_X}$ , the (reduced) freeness of the framed graph von Neumann algebra  $M_{G_X}$  is completely determined by the (reduced) freeness of the graph von Neumann algebra  $M_G$ , i.e., we can have that:

**Theorem 4.2.** *Let  $M_{G_X}$  be the framed graph von Neumann algebra of a framed graph  $G_X$ , and let  $D_{G_X}$  be the framed diagonal subalgebra of  $M_{G_X}$ . Let  $E_0$  be the conditional expectation defined in the previous paragraph. Then*

$$M_{G_X} \stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}_{D_{G_X}}^r M_{X,e},$$

with the  $D_{G_X}$ -free blocks  $M_{X:e}$ ,

$$M_{X:e} \stackrel{\text{def}}{=} vN(M_e, M_X) \stackrel{*-\text{iso}}{=} M_X \otimes_{\mathbb{C}} M_e, \text{ in } B(H_{G_X}),$$

for all  $e \in E(G)$ .

**Proof.** The framed graph von Neumann algebra  $M_{G_X}$  is  $*$ -isomorphic to the von Neumann algebra  $M_X \otimes_{\mathbb{C}} M_G$ , where  $M_X$  is the von Neumann algebra  $L^\infty(X, \mu)$ , and  $M_G$  is the usual graph von

Neumann algebra of  $G$ . Let's denote  $M_X \otimes_{\mathbb{C}} M_G$  and  $M_X \otimes_{\mathbb{C}} D_G$  by  $\mathcal{M}_{G_X}$  and  $\mathcal{D}_{G_X}$ , respectively, where  $D_G$  is the usual diagonal subalgebra of  $M_G$ .

Define now a conditional expectation

$$\mathcal{E}_0 : \mathcal{M}_{G_X} \rightarrow \mathcal{D}_{G_X}$$

by

$$\mathcal{E}_0 \stackrel{\text{def}}{=} i_d \otimes E,$$

where  $i_d$  means the identity map, and  $E$  is the canonical conditional expectation from  $M_G$  onto  $D_G$ , i.e.,  $\mathcal{E}_0$  is the linear map satisfying that

$$\mathcal{E}_0(g \otimes x) = g \otimes E(x), \text{ for all } g \otimes x \in \mathcal{M}_{G_X}.$$

Then, the amalgamated  $W^*$ -probability spaces  $(M_{G_X}, E_0)$  and  $(\mathcal{M}_{G_X}, \mathcal{E}_0)$  are equivalent in the sense of Voiculescu over  $D_{G_X} \stackrel{*-\text{iso}}{=} \mathcal{D}_{G_X}$ . This shows that the  $D_{G_X}$ -freeness on  $M_{G_X}$  (depending on  $E_0$ ), and the  $\mathcal{D}_{G_X}$ -freeness on  $\mathcal{M}_{G_X}$  (depending on  $\mathcal{E}_0$ ) are equivalent. So,

$$M_{G_X} \stackrel{*-\text{iso}}{=} \mathcal{M}_{G_X} \stackrel{\text{def}}{=} M_X \otimes_{\mathbb{C}} M_G \stackrel{*-\text{iso}}{=} M_X \otimes_{\mathbb{C}} \left( \begin{matrix} *^r_{D_G} \\ e \in E(G) \end{matrix} Me \right),$$

where  $M_e$ 's are the  $D_G$ -free blocks  $vN(\mathbb{G}_e)$ 's for all  $e \in E(G)$

$$\stackrel{*-\text{iso}}{=} \begin{matrix} *^r \\ M_X \otimes_{\mathbb{C}} D_G \\ e \in E(G) \end{matrix} vN(M_X, M_e)$$

$$\stackrel{*-\text{iso}}{=} \begin{matrix} *^r \\ D_{G_X} \\ e \in E(G) \end{matrix} M_X \otimes_{\mathbb{C}} M_e \quad (\text{in } B(H_{G_X}))$$

$$\stackrel{*-\text{iso}}{=} \begin{matrix} *^r \\ D_{G_X} \\ e \in E(G) \end{matrix} M_{X:e} \quad \blacksquare$$

The above theorem shows that the free probability is also used to study framed graph von Neumann algebras under certain conditions.

**4.3. Framed graph von Neumann algebras.** Let  $G$  be a directed graph with its graph groupoid  $\mathbb{G}$ ,  $X = (X, \mathcal{B}_X, \mu)$  be a given Borel space, and let  $M_X = L^\infty(X, \mu)$ . Let  $G_X$  be the framed graph of  $G$  with the frame  $X$ , and  $\mathbb{G}_X$  be the framed groupoid of  $G_X$ . Also, let  $M_{G_X} = \overline{\mathbb{C}[L(\mathbb{G}_X)]}^w$  be the framed graph von Neumann algebra in  $B(H_{G_X})$ .

Remember that  $M_G$  has the amalgamated ( $D_G$ -valued) reduced free structures;

$$M_G \stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}_{D_G}^r M_e,$$

with its  $D_G$ -free blocks  $M_e = vN(\mathbb{G}_e)$ , for all  $e \in E(G)$ , where  $\mathbb{G}_e$  are the subgroupoids of  $\mathbb{G}$ , consisting of all reduced words only in  $\{e, e^{-1}\}$ .

In the previous section, we defined the framed diagonal subalgebra  $D_{G_X}$ , and the conditional expectation  $E_0 : M_{G_X} \rightarrow D_{G_X}$ . We concluded the amalgamated freeness on the framed graph von Neumann algebra  $M_{G_X}$ ;

$$M_{G_X} \stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}_{D_{G_X}}^r M_{X:e},$$

where

$$M_{X:e} \stackrel{\text{def}}{=} vN(M_X, M_e) \stackrel{*-\text{iso}}{=} M_X \otimes_{\mathbb{C}} M_e,$$

for all  $e \in E(G)$ , where  $M_e$ 's are the  $D_G$ -free blocks of the graph von Neumann algebra  $M_G$  of  $G$ , i.e., the  $D_{G_X}$ -valued freeness is naturally determined on  $M_{G_X}$ , by the  $D_G$ -valued freeness of  $M_G$ , tensorized by  $M_X$ .

By using the  $*$ -isomorphic relation of the framed graph von Neumann algebra  $M_{G_X}$ , and  $\mathcal{M}_{G_X} = M_X \otimes_{\mathbb{C}} M_G$ , we define now a new conditional expectation  $E_X : M_{G_X} \rightarrow D_G$  by

$$E_X \stackrel{\text{def}}{=} (I_{nt} \otimes E) \circ \Phi,$$

i.e.,

$$E_X \left( \sum_{(B,w) \in \mathbb{G}_X} t_{(B,w)} L_{(B,w)} \right) = I_{nt} \left( \sum_{(B,w) \in \mathbb{G}_X} t_{(B,w)} \chi_B \right) E \left( \sum_{(B,w) \in \mathbb{G}_X} L_w \right),$$

where  $\Phi$  is a  $*$ -isomorphism between  $M_{G_X}$  and  $\mathcal{M}_{G_X}$ , and  $E$  is the canonical conditional expectation from  $M_G$  onto  $D_G$ , and where

$$I_{nt} : M_X \rightarrow \mathbb{C}$$

is the conditional expectation (in fact, it is a bounded linear functional) defined by

$$I_{nt}(g) \stackrel{\text{def}}{=} \int_X g \, d\mu, \text{ for all } g \in M_X.$$

We call  $E_X$  the *diagonal conditional expectation on  $M_{G_X}$* . With respect to the new conditional expectation  $E_X$ , we can obtain the following freeness condition on  $M_{G_X}$ , different from the Subsection 4.3 above.

**Theorem 4.3.** *Let  $M_{G_X}$  be a framed graph von Neumann algebra and let  $E_X : M_{G_X} \rightarrow D_G$  be the diagonal conditional expectation on  $M_{G_X}$ , where  $D_G$  is the diagonal subalgebra of the graph von Neumann algebra  $M_G$ . Then*

$$M_{G_X} \stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}_{D_G}^r M_{X:e},$$

where

$$M_{X:e} \stackrel{\text{def}}{=} vN(M_X, M_e) \stackrel{*-\text{iso}}{=} M_X \otimes_{\mathbb{C}} M_e,$$

for all  $e \in E(G)$ , where  $M_X = L^\infty(X, \mu)$ , and  $M_e$ 's are  $D_G$ -free blocks of  $M_G$ .

**Proof.** Let  $E_X : M_{G_X} \rightarrow D_G$  be the diagonal conditional expectation,

$$E_X = (I_{nt} \otimes E) \circ \Phi.$$

Then, we have

$$M_{G_X} \stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}^r_{D_{G_X}} M_{X:e},$$

with respect to the conditional expectation  $E_0$ , by Subsection 4.3

$$\stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}^r_{M_X \otimes_{\mathbb{C}} D_G} M_X \otimes_{\mathbb{C}} M_e$$

$$\stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}^r_{D_G} M_X \otimes M_e,$$

with respect to the conditional expectation  $I_{nt}$

$$\stackrel{*-\text{iso}}{=} \underset{e \in E(G)}{*}^r_{D_G} M_{X:e}. \quad \blacksquare$$

**4.4. Calculus on  $X$  affected by graph groupoids.** Let  $X = (X, \mathcal{B}_X, \mu)$  be a given Borel space and let  $G$  be a directed graph with its graph groupoid  $\mathbb{G}$ . For a given Borel space  $X$ , we can naturally determine the calculus on  $X$ ; for any  $B \in \mathcal{B}_X$ , we have the integrals of the characteristic functions  $\mathcal{X}_B$ ,

$$\int_X \mathcal{X}_B d\mu = \mu(B).$$

Recall that all elements  $g$  of  $M_X = L^\infty(X, \mu)$  are approximated by the simple functions formed by

$$\sum_{B \in \mathcal{B}_X} t_B \chi_B, \text{ with } t_B \in \mathbb{C}.$$

Thus, the integrals  $\int_X g \, d\mu$  are well-defined, and hence we have calculus on  $X$ .

We extend such an integration on  $X$  to multi-dimensional integration on the framed graph von Neumann algebra  $M_{G_X}$ . Then, this will show how the calculus on  $X$  is affected by the (outside) structures, represented by the graph groupoids  $\mathbb{G}$  of graphs  $G$ . Notice that the changes of the calculus on  $X$  from  $\mathbb{G}$  becomes not only be multi-dimensional, but also be dictated by the admissibility on  $\mathbb{G}$ .

Such changes are explained by the conditional expectation  $E_X = (I_{nt} \otimes E) \circ \Phi$ , where  $I_{nt} : M_X \rightarrow \mathbb{C}$  is the integration on  $X$ , and  $E : M_G \rightarrow D_G$  is the canonical conditional expectation introduced in Subsection 3.3, and  $\Phi$  is a  $*$ -isomorphism between  $M_{G_X}$  and  $M_X \otimes_{\mathbb{C}} M_G$ .

Let  $a \in M_{G_X}$  with its expression,

$$a = \sum_{(B,w) \in \mathbb{G}_X} t_{(B,w)} L_{(B,w)}.$$

Then the *integral of  $a$* , as a  $\mathbb{C}^{\oplus |V(G)|}$ -value (recall that  $D_G \stackrel{*-\text{iso}}{=} \mathbb{C}^{\oplus |V(G)|}$ ), can be defined by the expectation  $E_X(a)$ , i.e.,

$$\begin{aligned} E_X(a) &= (I_{nt} \otimes E) \left( \left( \sum_{B \in \mathcal{B}_X, t_{(B,w)} \neq 0} t_{(B,w)} \chi_B \right) \otimes \left( \sum_{w \in \mathbb{G}, t_{(B,w)} \neq 0} L_w \right) \right) \\ &= \left( \int_X \sum_{B \in \mathcal{B}_X, t_{(B,w)} \neq 0} t_{(B,w)} \chi_B \, d\mu \right) \left( E \left( \sum_{w \in \mathbb{G}, t_{(B,w)} \neq 0} L_w \right) \right) \end{aligned}$$

$$= \left( \sum_{B \in \mathcal{B}_X, t(B,w) \neq 0} t_{(B,w)} \mu(B) \right) \left( \sum_{v \in V(G), t(B,v) \neq 0} L_v \right). \tag{4.4.1}$$

Denote the multipliers

$$\sum_{B \in \mathcal{B}_X, t(B,w) \neq 0} t_{(B,w)} \mu(B), \text{ and } \sum_{v \in V(G), t(B,v) \neq 0} L_v$$

of (4.4.1), by  $I_a$  and  $E_a$ , respectively. We can realize that, if

$$g = \sum_{B \in \mathcal{B}_X, t(B,w) \neq 0} t_{(B,w)} \chi_B \in M_X,$$

with its integral

$$\int_X g \, d\mu = I_a,$$

then such an integral  $\int_X g \, d\mu$  of  $g$  is changed (in  $M_{G_X}$ ) by the expectation  $E_a$  of

$$\sum_{w \in \mathbb{G}, t(B,w) \neq 0} L_w \in M_G,$$

dependent upon the elements  $w$  of the graph groupoid  $\mathbb{G}$  satisfying that  $t(B,w) \neq 0$ .

**Observation.** The above discussion shows that the  $\mathbb{C}^{\oplus|V(G)|}$ -valued integration on the von Neumann algebra  $M_{G_X}$  is well-defined by the diagonal conditional expectation  $E_X$ , and it is dependent upon the integration  $I_{nt} = \int_X \cdot \, d\mu$  on  $X$ , and the admissibility on the graph groupoid  $\mathbb{G}$ . This means that the integration  $I_{nt}$  on  $X$  is affected by the dynamics of  $\mathbb{C}$  determined by  $\mathbb{G}$ . □

### 5. Fractal Graphs

In the previous section, we consider (i) how the calculus on a Borel space  $X = (X, \mathcal{B}_X, \mu)$  is extended to the calculus on a certain operator algebra  $B(H_{G_X})$  (in particular, on a von Neumann algebra  $M_{G_X}$ ), and (ii) how the obstacles  $\mathbb{G}$  (certain operators, in particular, partial isometries  $\mathbb{G}$  embedded in  $L(\mathbb{G}_X)$ ) in the von Neumann algebra  $M_{G_X}$  changes the integration  $I_{nt} = \int_X \cdot d\mu$  on  $X$  in the operator-valued integration  $E_X : M_{G_X} \rightarrow D_G$ .

To find the concrete examples for applications, we consider a special kind of graphs, and their graph groupoids. We would like to handle “good” graphs having certain regularity, represented by the *fractal property*, in short, the *fractality*. In this section, we will concentrate on introducing *fractal graphs* and their graph groupoids, called the *graph fractaloids*. Roughly speaking, fractal graphs are the graphs generating the fractality in their graph groupoids. The study of fractal graphs and graph fractaloids, itself, is interesting (see [9], [12], [15], [16], [19], and [21]).

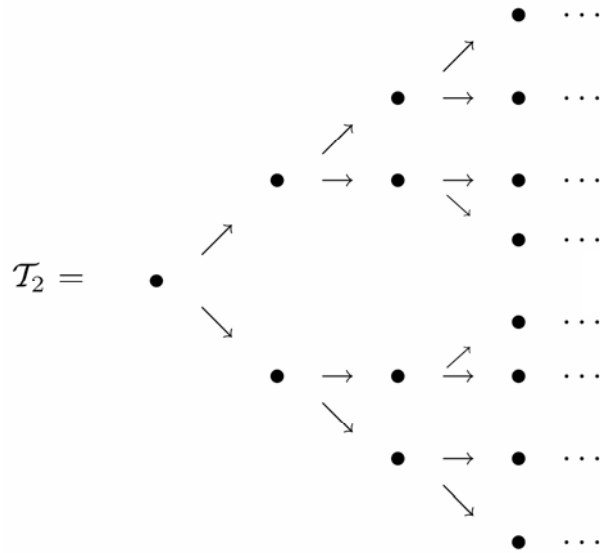
Remember that all our graphs are assumed to be connected and locally finite. (Of course, the results of Section 4 hold for the general cases. Hence, we did not consider / mention this assumption much, but from now, this assumption is very much needed!) Recall that a graph  $G$  is *connected*, if for any pair  $(v, v')$  of “distinct” vertices, there always exists a reduced finite path  $w$  in the graph groupoid  $\mathbb{G}$  of  $G$ , such that  $w = v w v'$ , and  $w^{-1} = v' w^{-1} v$ . Also, a graph  $G$  is said to be *locally finite*, if all vertices  $v$  of  $G$  has finite degrees, i.e.,  $\deg(v) < \infty$ .

In this section, we concentrate on introducing fractal graphs, graph fractaloids, and their basic properties. The definition of fractal graphs is based on the connectedness and the locally-finiteness.



**5.1. Graph trees.** In this section, we construct the graph tree  $\mathcal{T}_G$  induced by a given connected locally finite directed graph  $G$ . Throughout this section, all graphs are automatically assumed to be connected, and locally finite.

Recall that a directed graph, having neither multi-edges nor loop finite paths, is called a *directed tree*. If a directed tree  $G$  has at least one vertex  $v$ , satisfying that  $\text{deg}_{in}(v) = 0$ , is said to be a *directed tree with root(s)*. The vertices with 0 in-degree are called the roots of  $G$ . Suppose, we have a directed tree  $G$  with roots, and assume that we fix one root  $v_0$ . Then  $G$  is called a *rooted tree* with its root  $v_0$ . Now, let  $G$  be a rooted tree with its root  $v_0$ , and assume that the direction of  $G$  is one-flow from the root  $v_0$ . Then  $G$  is a *one-flow rooted tree*. A one-flow rooted tree is infinite, then it is said to be a *growing rooted tree*. Assume that a growing rooted tree  $G$  satisfies that, for any  $v \in V(G)$ , the out-degree  $\text{deg}_{out}(v)$  are all identical. Then  $G$  is a *regular tree*. In particular, if  $\text{deg}_{out}(v) = N$ , for all  $v \in V(G)$ , then this regular tree  $G$  is called the  *$N$ -regular tree*. To emphasize the regularity of this tree  $G$ , we denote this  $N$ -regular tree  $G$  by  $\mathcal{T}_N$ . For instance, the 2-regular tree  $\mathcal{T}_2$  is as follows:



Let  $G$  be a graph, and let

$$N = \max\{\text{deg}_{out}(v) : v \in V(G)\} < \infty \text{ in } \mathbb{N}.$$

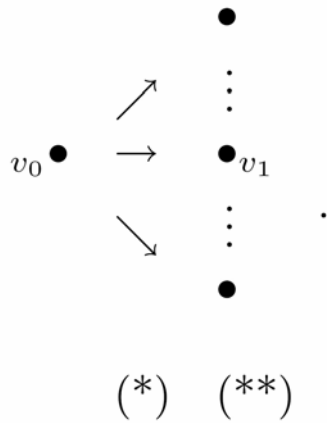
Consider the shadowed graph  $\widehat{G}$  of  $G$ . Define the subsets  $E_v^{v'}$  of  $E(\widehat{G})$  by

$$E_v^{v'} \stackrel{\text{def}}{=} \{e \in E(\widehat{G}) : e = v e v'\},$$

for all  $(v, v') \in V(\widehat{G})^2$ . Remark that  $v$  and  $v'$  are not necessarily distinct in  $V(\widehat{G})$ . It is possible that there exists a pair  $(v_1, v_2)$  of vertices such that  $E_{v_1}^{v_2}$  is empty. By definition,

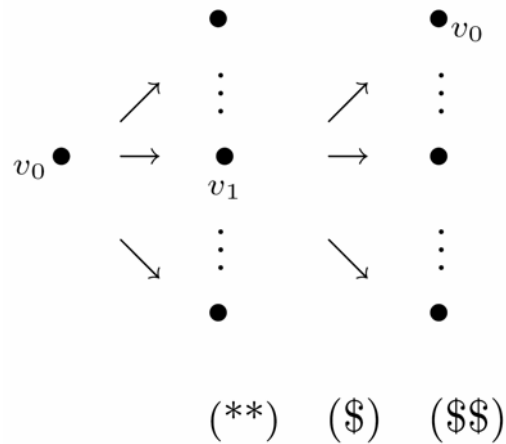
$$E(\widehat{G}) = \bigcup_{(v, v')} E_v^{v'}.$$

Then construct the graph tree  $\mathcal{T}_G$  of  $G$ , by re-arranging the elements  $V(\widehat{G}) \cup E(\widehat{G})$ , up to the admissibility on the free semigroupoid  $\mathbb{F}^+(\widehat{G})$ , as follows. First, fix any arbitrary vertex  $v_0 \in V(\widehat{G}) = V(G)$ . Then arrange  $e \in \bigcup_{v \in V(\widehat{G})} E_{v_0}^v$ , by attaching them to  $v_0$ , preserving the direction on  $G$ , i.e., we can construct



Then, we can have the above finite rooted tree with its root  $v_0$ . Of course, if the set  $\bigcup_{v \in V(\widehat{G})} E_{v_0}^v$  is empty, then we only have the trivial tree  $G_{v_0}$ , with  $V(G_{v_0}) = \{v_0\}$ , and  $E(G_{v_0}) = \emptyset$ . The edges in the column (\*) is induced by the re-arrangement of the elements in  $\bigcup_{v \in V(\widehat{G})} E_{v_0}^v$ , and the vertices in the column (\*\* ) means the re-arrangement of the “terminal” vertices of the edges in  $\bigcup_{v \in V(\widehat{G})} E_{v_0}^v$ .

Now, let  $v_1 \in V(\widehat{G})$  be an arbitrary chosen vertex of the shadowed graph  $\widehat{G}$  of  $G$ , re-arranged in (\*\*). Then, we can do the same process for  $v_1$ , i.e., arrange the edges in  $\bigcup_{v \in V(\widehat{G})} E_{v_1}^v$  (if it is not empty), by attaching them to  $v_1$ , preserving the direction on  $G$ , i.e., we can construct



Here, the column (\$) is induced by the re-arrangement of the edges in  $\bigcup_{v \in V(\widehat{G})} E_{v_1}^v$ , and the vertices in the column (\$\$) means the re-arrangement of the terminal vertices of the edges in  $\bigcup_{v \in V(\widehat{G})} E_{v_1}^v$ . We can

do the same processes for all vertices in (\*\*). Now, notice that it is possible that one of the vertices in the columns (\*\*) or (§§) can be  $v_0$ . For instance, if  $E_{v_0}^{v_0}$  is not empty (equivalently, if  $v_0$  has an incident loop-edge), then  $v_0$  is located in (\*\*). Similarly,  $v_0$  can be located in (§§). For instance, if  $v_0$  has its incident length-2 loop finite path in  $\mathbb{F}^+(\widehat{G})$ , then  $v_0$  is in (§§). We admit such cases, i.e., a same vertex of  $V(\widehat{G})$  can appear several times in this rooted-tree-making process.

Do this process until it ends. If  $G$  is infinite, then do this process infinitely. The one-flow rooted tree, induced by this process, with its root  $v_0$  is denoted by  $\mathcal{T}_{v_0}$ . Notice that, from this process, we can embed all elements (possibly several or infinitely many repeated times) in  $V(\widehat{G}) \cup E(\widehat{G})$  into  $\mathcal{T}_{v_0}$ , preserving their admissibility!

**Definition 5.1.** Let  $G$  be a connected locally finite directed graph with its shadowed graph  $\widehat{G}$ , and let  $\mathcal{T}_{v_0}$  be a rooted tree with its root  $v_0$ , induced by  $G$ , by the above process. We say that, this process the graph-tree making, and the tree  $\mathcal{T}_{v_0}$  is called the  $v_0$ -tree of  $G$ .

By definition, every connected locally finite directed graph  $G$  has  $|V(\widehat{G})|$ -many vertex-trees of  $G$ . Notice that, the vertex-trees of  $G$  are determined by the vertices and edges in the “shadowed” graph  $\widehat{G}$  of  $G$ . The following proposition is easily proven by the definition of the vertex-trees of a given graph, and by the connectedness of our graphs.

Observe now several examples for the construction of vertex-graphs of a given graph.

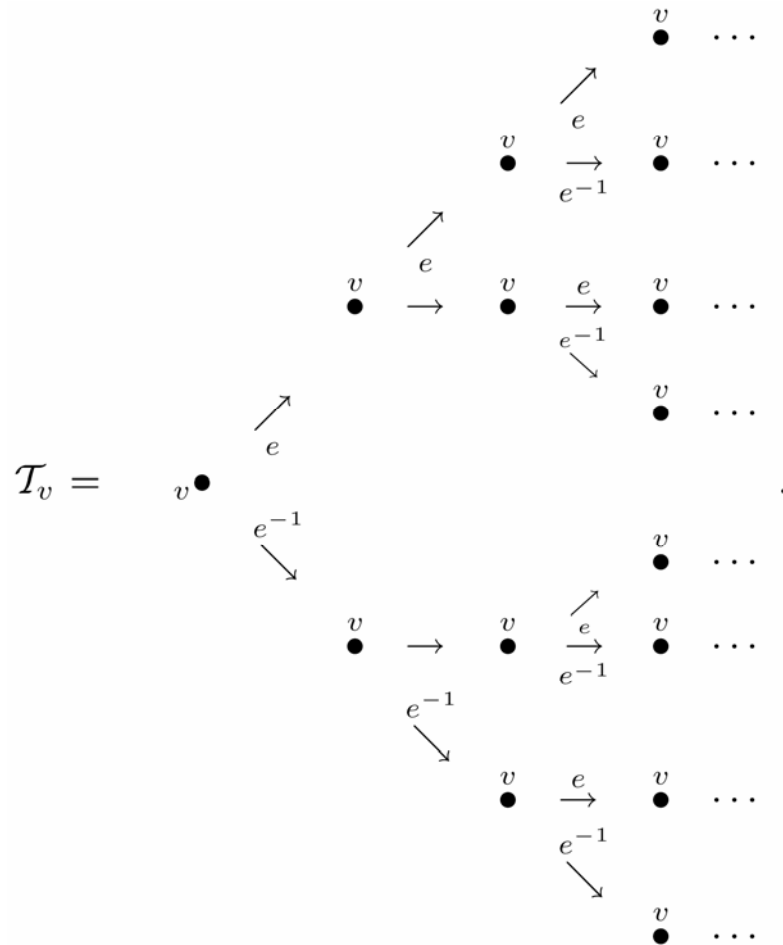
**Example 5.1.** Let  $O_1$  be a one-vertex-1-loop-edge graph with

$$V(O_1) = \{v\}, \text{ and } E(O_1) = \{e = v e v\}.$$

Then, the shadowed graph  $\widehat{O}_1$  of  $O_1$  has its vertex set  $V(\widehat{O}_1)$ , identical to  $V(O_1)$ , and its edge set

$$V(\widehat{O}_1) = \{v\}, \text{ and } E(\widehat{O}_1) = \{e, e^{-1}\}.$$

Then, we can construct the  $v$ -graph of  $O_1$  by



We can realize that the  $v$ -graph  $T_v$  is graph-isomorphic to the 2-regular graph  $T_2$ .

**Example 5.2.** Let  $G_e$  be the two-vertices-one-edge graph with

$$V(G_e) = \{v_1, v_2\}, \text{ and } E(G_e) = \{e = v_1 e v_2\}.$$

Then, the shadowed graph  $\widehat{G_e}$  is a directed graph with

$$V(\widehat{G_e}) = \{v_1, v_2\}, \text{ and } E(\widehat{G_e}) = \{e, e^{-1}\}.$$

So, we can have the  $v_1$ -tree  $\mathcal{T}_{v_1}$  of  $G$ ,

$$\mathcal{T}_{v_1} = v_1 \bullet \xrightarrow{e} v_2 \bullet \xrightarrow{e^{-1}} v_1 \bullet \xrightarrow{e} v_2 \bullet \xrightarrow{e^{-1}} \dots,$$

and the  $v_2$ -tree  $\mathcal{T}_{v_2}$  of  $G$ ,

$$\mathcal{T}_{v_2} = v_2 \bullet \xrightarrow{e^{-1}} v_1 \bullet \xrightarrow{e} v_2 \bullet \xrightarrow{e^{-1}} v_1 \bullet \xrightarrow{e} \dots.$$

Therefore, both  $\mathcal{T}_{v_1}$  and  $\mathcal{T}_{v_2}$  are graph-isomorphic to the 1-regular tree  $\mathcal{T}_1$ .

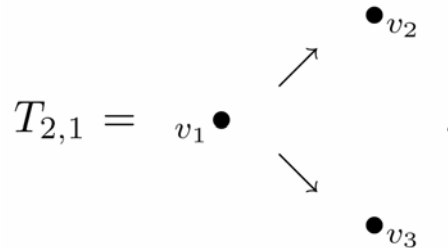
**Example 5.3.** Let  $T_{2,1}$  be the finite tree with

$$V(T_{2,1}) = \{v_1, v_2, v_3\},$$

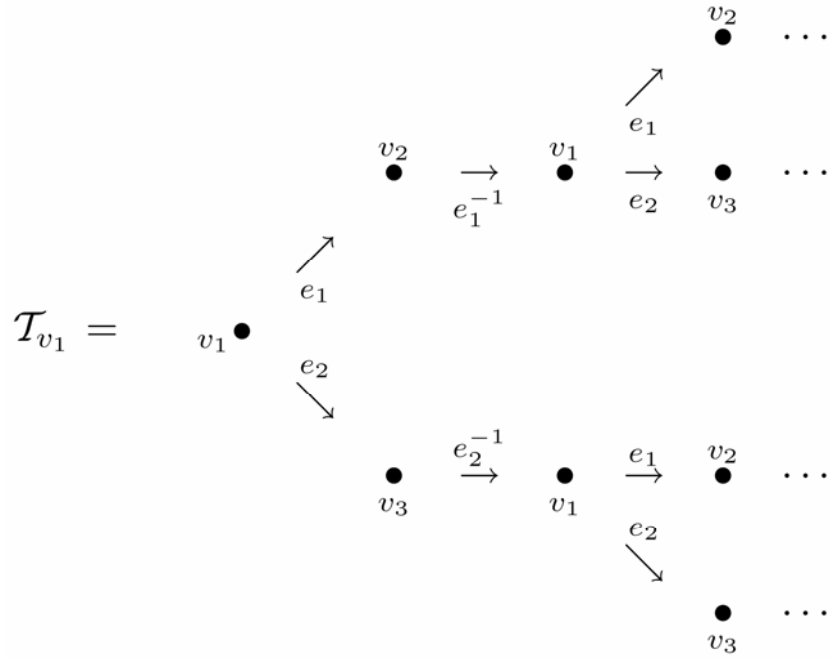
and

$$E(T_{2,1}) = \{e_1 = v_1 e_1 v_2, e_2 = v_1 e_2 v_3\},$$

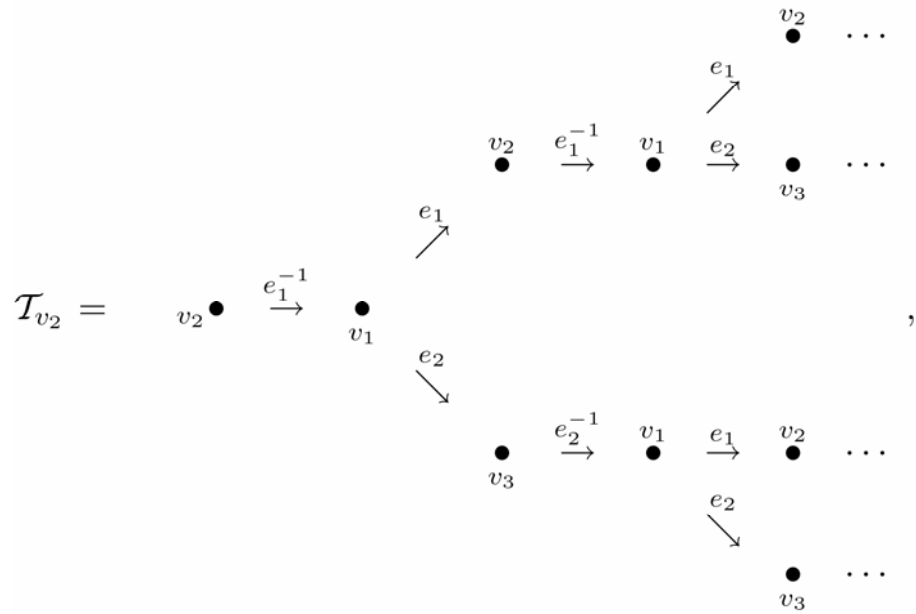
i.e.,



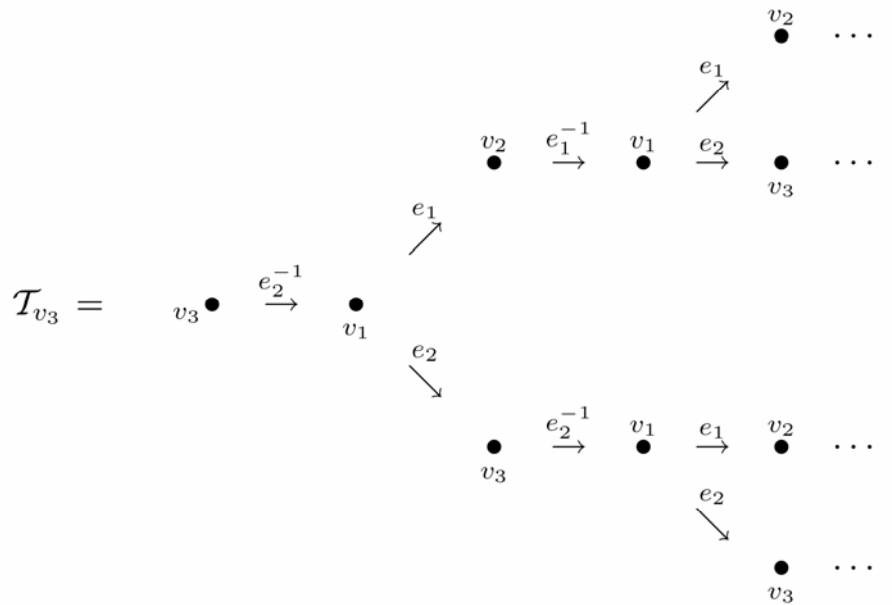
Then, after finding, the shadowed graph  $\widehat{T_{2,1}}$  of  $T_{2,1}$ , we can have the  $v_1$ -tree  $\mathcal{T}_{v_1}$  of  $T_{2,1}$ ,



and the  $v_2$ -tree  $\mathcal{T}_{v_2}$  of  $T_{2,1}$ ,



and the  $v_3$ -graph  $\mathcal{T}_{v_3}$  of  $T_{2,1}$ ,



We can check that  $\mathcal{T}_{v_2}$  and  $\mathcal{T}_{v_3}$  are graph-isomorphic, but neither of them is graph-isomorphic to  $\mathcal{T}_{v_1}$ .

**Example 5.4.** Let  $K_2$  be the one-flow circulant graph with

$$V(K_2) = \{v_1, v_2\},$$

and

$$E(K_2) = \{e_1 = v_1 e_1 v_2, e_2 = v_2 e_2 v_1\}.$$

Then, the shadowed graph  $\widehat{K_2}$  of  $K_2$  has

$$V(\widehat{K_2}) = \{v_1, v_2\}, \text{ and } E(\widehat{K_2}) = \{e_1^{\pm 1}, e_2^{\pm 1}\}.$$

By using the tree-making process, we obtain that the  $v_1$ -graph  $\mathcal{T}_{v_1}$  and the  $v_2$ -graph  $\mathcal{T}_{v_2}$  are graph-isomorphic to the 2-regular tree  $\mathcal{T}_2$ .



**Example 5.5.** We say that  $C_n$  is the complete graph with  $n$ -vertices, if for any pair  $(v, v')$  of distinct vertices in  $V(C_n)$ , there exists a unique edge  $e \in E(C_n)$ , such that  $e = v e v'$ , where  $n \in \mathbb{N} \setminus \{1\}$ . We can easily check that  $\text{deg}_{out}(v) = n - 1$ , for all  $v \in V(C_n)$ . Thus, we can check that the graph trees  $\mathcal{T}_v$  are graph-isomorphic to the  $2(n - 1)$ -regular tree  $\mathcal{T}_{2(n-1)}$ , for all  $v \in V(C_n)$ .

**Example 5.6.** Let  $\mathcal{T}_N$  be the  $N$ -regular tree, for  $N \in \mathbb{N}$ . Assume that  $v_0$  is the root of  $\mathcal{T}_N$ . It is easy to check that the  $v$ -trees  $\mathcal{T}_v$  are graph-isomorphic to the  $2N$ -regular tree  $\mathcal{T}_{2N}$ , whenever  $v \neq v_0$  in  $V(\mathcal{T}_N)$ . However, the  $v_0$ -tree  $\mathcal{T}_{v_0}$  is not graph-isomorphic to  $\mathcal{T}_{2N}$ .

As we have seen in the previous examples, sometimes, the vertex-trees of a given graph are graph-isomorphic from each other, or not. In general, the vertex-trees of a graph  $G$  are not graph-isomorphic from each other.

**5.2. Fractal graphs.** Let  $G$  be a connected, locally finite directed graph with its graph groupoid  $\mathbb{G}$ . By Subsection 5.1, for the given graph  $G$ , we can construct the vertex-fixed graph trees  $\{\mathcal{T}_v : v \in V(G)\}$  of  $G$ . Define the graph fractaloids and the fractal graphs.

**Definition 5.2.** Let  $G$  be a connected locally finite directed graph and  $\{\mathcal{T}_v : v \in V(\widehat{G})\}$ , the collection of all vertex-trees of  $G$ . Also, let

$$N = \max \{ \text{deg}_{out}(v) : v \in V(G) \} \text{ in } G,$$

where  $\text{deg}_{out}(\cdot)$  means the out-degree of vertices. If every  $v$ -tree  $\mathcal{T}_v$  of  $G$  is graph-isomorphic to the  $2N$ -regular tree  $\mathcal{T}_{2N}$ , for all  $v \in V(\widehat{G})$ , then the graph groupoid  $\mathbb{G}$  of  $G$  is called the *graph fractaloid*. Also, we call the graph  $G$ , a *fractal graph*, i.e., a connected locally finite directed graph, generating a graph fractaloid, is said to be a *fractal graph*.

In [15] and [16], we define graph fractaloids by the *labelling process* on graph groupoids, determined by *automata theory*. Automata theory let us detect the fractality on graph groupoids. Even though our definition of graph fractaloids are defined without automata settings, it is well-defined in the same sense of [15] and [16], because our vertex-fixed graph trees, in the sense of Subsection 5.1, are equivalent version of automata trees, in the sense of [15].

Now, we introduce the characterizations of graph fractaloids of [15].

**Theorem 5.1** (see [15]). *Let  $G$  be a connected locally finite directed graph with its graph groupoid  $\mathbb{G}$ , where  $N$  is the maximum of the out-degrees of the vertices of  $G$ . And let  $\mathcal{A}_G$  be the graph automaton in the sense of [15], having the corresponding automata actions  $\{\mathcal{A}_w : w \in \mathbb{F}^+(\widehat{G})\}$ .*

(1)  $\mathbb{G}$  is a graph fractaloid, if and only if the automata actions act fully on the  $2N$ -regular tree  $T_{2N}$ .

(2)  $\mathbb{G}$  is a graph fractaloid, if and only if the all automata trees, induced by the automata actions  $\mathcal{A}_w$ 's, are graph-isomorphic to the  $2N$ -regular tree  $T_{2N}$ . □

The statement (1) of the previous theorem provides the automata-theoretical characterization of graph fractaloids, and the statement (2) provides the algebraic characterization of graph fractaloids. These two characterizations show that indeed our vertex-fixed graph trees in the sense of Subsection 5.1 are the graph-theoretical re-expression of the automata trees of graph groupoids in the sense of [15]. Thanks to these characterizations, we found the pure graph-theoretical characterization of graph fractaloids in [21].

**Theorem 5.2** (see [21]). *Let  $G$  be a connected locally finite directed graph with*

$$N = \max \{ \deg_{out}(v) : v \in V(G) \}, \text{ in } G.$$

Then, the graph  $G$  is a fractal graph, if and only if

$$\deg_{out}(v) = N = \deg_{in}(v), \text{ for all } v \in V(G). \quad \square$$

The above characterization gives the best way to detect the fractality of graph groupoids, simply by checking the out-degrees and in-degrees of vertices of a given graph. It is interesting that the fractality on an algebraic structure, graph groupoids, is detected by the pure combinatorial data, degrees of vertices.

The above graph-theoretical characterization of graph fractaloids has its equivalent version.

**Theorem 5.3** (see [21]). *Let  $G$  be a connected locally finite directed graph with*

$$N = \max \{ \deg_{out}(v) : v \in V(G) \}, \text{ in } G.$$

Then, the graph  $G$  is a fractal graph, if and only if

$$\deg_{out}^{(\widehat{G})}(v) = 2N, \text{ in } \widehat{G},$$

for all  $v \in V(\widehat{G}) = V(G)$ , where  $\widehat{G}$  is the shadowed graph of  $G$ , and  $\deg_{out}^{(\widehat{G})}(\cdot)$  means the out-degree of vertices of  $\widehat{G}$ . □

By the previous two theorems, we can have the following easy tools to detect the fractality of graph groupoids.

**Proposition 5.4.** *Let  $G$  be a connected locally finite directed graph.*

- (1) *If there is a vertex  $v_0$  of  $G$ , such that  $\deg_{out}(v_0) \neq \deg_{in}(v_0)$  in  $G$ , then  $G$  is not fractal.*
- (2) *If there is a pair  $(v, v')$  of vertices, such that  $\deg_{k_1}(v) \neq \deg_{k_2}(v')$ , for some  $k_1, k_2 \in \{in, out\}$ , then  $G$  is not fractal.*
- (3) *If  $G$  contains either a sink or a source, then  $G$  is not fractal.* □

Recall that a vertex  $v \in V(G)$  is a *sink* (or a *source*), if  $\deg_{out}(v) = 0$  (resp.,  $\deg_{in}(v) = 0$ ). The statement (3) of the previous proposition shows that any regular trees are not fractal, since the roots of the trees are sources.

**Example 5.7.** (1) The one-vertex- $n$ -loop-edge graph  $O_n$ , with

$$V(O_n) = \{v\},$$

and

$$E(O_n) = \{e_j = v e_j v : j = 1, \dots, n\}$$

is fractal, for all  $n \in \mathbb{N}$ , since

$$\deg_{out}(v) = n = \deg_{in}(v), \text{ in } O_n,$$

where  $v$  is the unique vertex of  $O_n$ .

(2) The one-flow circulant graph  $K_m$ , with

$$V(K_m) = \{v_1, \dots, v_m\},$$

and

$$E(K_m) = \{e_j = v_j e_j v_{j+1} : j = 1, \dots, m, \text{ with } v_{m+1} \stackrel{\text{def}}{=} v_1\}$$

is fractal, for all  $m \in \mathbb{N} \setminus \{1\}$ , since

$$\deg_{out}(v_j) = 1 = \deg_{in}(v_j), \text{ in } K_m,$$

for all  $j = 1, \dots, m$ .

(3) Let  $C_m$  be the complete graph with

$$V(C_m) = \{v_1, \dots, v_m\},$$

and

$$E(C_m) = \{e_{ij} : i \neq j \in \{1, \dots, m\}\},$$

where  $e_{ij}$  means the edge connecting the vertex  $v_i$  to the vertex  $v_j$ , for  $i, j \in \{1, \dots, m\}$ , for  $m \in \mathbb{N} \setminus \{1\}$ . Then  $C_m$  is a fractal graph, since

$$\text{deg}_{out}(v_j) = n - 1 = \text{deg}_{in}(v_j), \text{ in } C_m,$$

for all  $j = 1, \dots, m$ .

(4) The infinite linear graph  $L$ , graph-isomorphic to

$$\dots \rightarrow \bullet \rightarrow \bullet \rightarrow \dots$$

is a fractal graph, since

$$\text{deg}_{out}(v) = 1 = \text{deg}_{in}(v),$$

for all  $v \in V(L)$ .

For more interesting example, see [10].

**5.3. Radial operators of graph fractaloids.** Let  $G$  be a fractal graph with its graph fractaloid  $\mathbb{G}$ , and let  $M_G$  be the graph von Neumann algebra of  $G$ . Let  $T_G \in M_G$  be the radial operator in the sense of Subsection 3.4. In [15], we consider the spectral information of  $T_G$ , by computing the  $D_G$ -valued free moments  $\{E(T_G^n)\}_{n=1}^\infty$ , where  $D_G$  is the diagonal subalgebra of  $M_G$ , and  $E$  is the canonical conditional expectation from  $M_G$  onto  $D_G$ . Since  $T_G$  is self-adjoint in  $M_G$ , the free distributional data represents the spectral information of  $T_G$ . So, instead of observing the spectral data of  $T_G$ , we computed free moments of it. The following theorem is the main result of [15].

**Theorem 5.5** (see [15]). *Let  $G$  be a fractal graph, and  $T_G \in M_G$ , the radial operator of the graph fractaloid  $\mathbb{G}$  of  $G$ . Then, the  $D_G$ -valued free moments are*

$$E(T_G^n) = |\mathcal{L}_N^o(n)| \cdot 1_{D_G},$$

for all  $n \in \mathbb{N}$ , where  $\mathcal{L}_N^o(n)$  is given in Appendix. □

It is interesting that the operator-valued (or amalgamated) free moments of  $T_G$  is completely determined by the scalar-values  $\{|\mathcal{L}_N^o(n)|\}_{n=1}^\infty$ .

**5.4. Richness of graph fractaloids.** In this section, we briefly consider the mathematical richness of graph fractaloids. There are “sufficiently” many graph fractaloids, induced by connected locally finite (even finite) graphs (see [10]). Moreover, in [10], we showed that, whenever we choose a pair  $(n, m) \in \mathbb{N} \times \mathbb{N}_\infty$ , where  $\mathbb{N}_\infty \stackrel{\text{def}}{=} \mathbb{N} \cup \{\infty\}$ , there always exists at least one fractal graph  $G$ .

**Theorem 5.6** (see [10]). *Let  $\mathcal{F}_{ractal}$  be the set of all connected locally finite fractal graphs. If*

$$[(n, m)] \stackrel{\text{def}}{=} \left\{ \mathbb{G} \in \mathcal{F}_{ractal} \left| \begin{array}{l} \text{deg}_{out}(v) = n, \text{ in } G, \\ \forall v \in V(G), \text{ and} \\ m = |V(G)| \end{array} \right. \right\}$$

is a subset of  $\mathcal{F}_{ractal}$ , for all  $(n, m) \in \mathbb{N} \times \mathbb{N}_0$ , then all subsets  $[(n, m)]$  of  $\mathcal{F}_{ractal}$  are nonempty, and

$$\mathcal{F}_{ractal} \stackrel{\text{def}}{=} \bigsqcup_{(n,m) \in \mathbb{N} \times \mathbb{N}_0} ([(n, m)]). \quad \square$$

Notice that the subsets  $[(n, m)]$  of  $\mathcal{F}_{ractal}$  are in fact, the equivalence classes in  $\mathcal{F}_{ractal}$ , i.e., if we define an equivalence relation  $\mathcal{R}$  on  $\mathcal{F}_{ractal}$  by

$$G_1 \mathcal{R} G_2 \stackrel{\text{def}}{\Leftrightarrow} E(T_{G_1}^n) = E(T_{G_2}^n), \text{ for all } n \in \mathbb{N},$$

then the relation  $\mathcal{R}$  is an equivalence relation, and hence  $[(n, m)]$ 's are the equivalence classes in  $\mathcal{F}_{ractal}$ , where  $T_{G_k}$  are the radial operators of the graph fractaloids  $\mathbb{G}_k$  of  $G_k$ , for  $k = 1, 2$ . For any graph fractaloid  $\mathbb{G} \in [(n, m)]$ , induced by a fractal graph  $G$ , we have that

$$E(T_G^k) = |\mathcal{L}_n^o(k)| \cdot 1_{\mathbb{C}^{\oplus m}}, \text{ for all } k \in \mathbb{N}.$$

The equivalence relation  $\mathcal{R}$  is called the *spectral relation*, and the equivalence classes are called the *spectral classes*. Also, the classification in the above theorem is said to be the *spectral classification of graph fractaloids* (see [10]).

Notice that the free groups, which are “fractal groups” (see [39]), are contained in the spectral class  $[(n, 1)]$ . In general, if  $m > 1$ , then the elements in  $[(n, m)]$  are groupoids with fractality, which are not fractal groups.

### 6. Framed Fractal Graphs

Throughout this section, all graphs are connected and locally finite. Let  $G$  be a fractal graph with its graph fractaloid  $\mathbb{G}$ . As in Section 4, we can construct the framed graph  $G_X$ , with the frame  $X = (X, \mathcal{B}_X, \mu)$ , a Borel measure space, and the corresponding framed groupoid  $\mathbb{G}_X$ . By the graph-theoretical characterization of graph fractaloids, the given graph  $G$  satisfies

$$\text{deg}_{out}(v) = N = \text{deg}_{in}(v), \text{ in } G,$$

for all  $v \in V(G)$ , where

$$N = \max \{ \text{deg}_{out}(v) : v \in V(G) \} \in \mathbb{N}, \text{ in } G.$$

It is a natural question: How can we establish the fractality on the framed groupoid  $\mathbb{G}_X$ ?

Let  $G$  be a connected locally finite graph and let  $X = (X, \mathcal{B}_X, \mu)$  be a Borel measure space. Let  $G_X$  be the framed graph of  $G$  with the frame  $X$ , and let  $\mathbb{G}_X$  be the framed groupoid of  $G_X$ .

The fractality of  $\mathbb{G}_X$  may not be detected by the same tool, like in Section 5 (or, like in [15]), because we “framed” the graph  $G$  (or the groupoid  $\mathbb{G}$ ) with the Borel measure space  $X = (X, \mathcal{B}_X, \mu)$ . However, remark that the algebraic properties of the framed groupoid  $\mathbb{G}_X$  is completely dependent upon those of the graph fractaloid  $\mathbb{G}$ . So, we may / can extend our fractality to that of framed groupoids. The definition of framed fractaloids may seem artificial, but it is reasonable.

**Definition 6.1.** Let  $G$  be a connected locally finite graph with its graph groupoid  $\mathbb{G}$ . Let  $G_X$  be the framed graph of  $G$  with the frame  $X = (X, \mathcal{B}_X, \mu)$ . The framed graph  $G_X$  is said to be a *framed fractal graph*, if  $G$  is a fractal graph. Also, the framed groupoid  $\mathbb{G}_X$  of  $G_X$  is called the *framed (graph) fractaloid* of  $G$  with the frame  $X$  (or, of  $G_X$ ).

Roughly speaking, a *framed fractal graph*  $G_X$  is the topological space, generated by the directions (or the admissibility) of the given graph  $G$ , which satisfies the fractality.

In Section 6, we observed that, if  $G$  is a connected, “finite” graph, then there always exists a finite fractal graph  $G_o$  such that  $G \leq G_o$ . So, we can obtain the following corollary.

**Corollary 6.1.** *Let  $G_X$  be a framed graph of a connected “finite” graph  $G$ , with the frame  $X$ , and let  $\mathbb{G}_X$  be the framed groupoid of  $G_X$ . Then, there always exists a framed fractaloid  $\mathbb{G}_X^o$ , with the same frame  $X$ , such that  $\mathbb{G}_X$  is a subgroupoid of  $\mathbb{G}_X^o$ .*

**Proof.** Let  $G$  be given. Then, there exists a fractal graph  $G_o$ , such that  $G \leq G_o$ . So, we obtain the groupoid-inclusion,  $\mathbb{G} \subseteq \mathbb{G}_o$ , where  $\mathbb{G}_o$  is the graph fractaloid of  $G_o$ . ■

Here, we want to emphasize that, even though the fractality on the framed fractaloid  $\mathbb{G}_X$  is determined by that of  $\mathbb{G}$ , the properties (dependent upon the fractality) of  $\mathbb{G}_X$  and those of  $\mathbb{G}$  are different. By the very definition, a graph fractaloid  $\mathbb{G}$  is a pure algebraic object having



the fractality, and a framed fractaloid  $\mathbb{G}_X$  is an algebraic, topological, and measure-theoretical object. Thus, our definition of framed fractaloids enlarges the study of the measure-preserving fractality to dynamical systems, i.e., our framed fractaloids will give the  $W^*$ -dynamical systems in a certain operator algebra, measure-preserving fractality!

Also, by Subsection 5.4, for any fixed frame  $X$ , we can have the following classification of framed fractaloids for the fixed frame.

**Corollary 6.2.** *Let  $\mathcal{F}_{ractal}^X$  be the set of all framed fractaloids of framed fractal graphs, with the fixed frame  $X$ , i.e.,*

$$\mathcal{F}_{ractal}^X \stackrel{\text{def}}{=} \{\mathbb{G}_X : \mathbb{G}_X \text{ is the framed fractaloid}\}.$$

*Then, for any  $(n, m) \in \mathbb{N} \times \mathbb{N}_\infty$ , there exists at least one fractaloid  $\mathbb{G}_X \in \mathcal{F}_{ractal}^X$ , such that  $\mathbb{G} \in [(n, m)] \subset \mathcal{F}_{ractal}$ , where  $\mathbb{G}$  is the graph fractaloid of  $G$ , whenever  $\mathbb{G}_X$  is the framed groupoid of the framed graph  $G_X$ . Moreover,*

$$\mathcal{F}_{ractal}^X = \bigsqcup_{(n,m) \in \mathbb{N} \times \mathbb{N}_\infty} [(n, m)]_X,$$

*where  $[(n, m)]_X$  is the equivalence class in  $\mathcal{F}_{ractal}^X$  defined by*

$$\mathbb{G}_X \in [(n, m)]_X \Leftrightarrow \mathbb{G} \in [(n, m)] \subset \mathcal{F}_{ractal},$$

*where  $\mathcal{F}_{ractal}$  and  $[(n, m)]$  are given in Subsection 5.4. □*

By the previous corollary, we can have the following corollary:

**Corollary 6.3.** *Let  $\mathcal{F}_{ractal}^{Frame}$  be the set of all framed groupoids of framed fractal graphs. Then*

$$\mathcal{F}_{ractal}^{Frame} = \bigsqcup_{X: \text{Bounded Borel Measure Space}} \left( \bigsqcup_{(n,m) \in \mathbb{N} \times \mathbb{N}_\infty} [(n, m)]_X \right).$$

□

## 7. Framed Radial Operators

Let  $G$  be a connected locally finite graph and  $X = (X, \mathcal{B}_X, \mu)$  be a Borel measure space, and let  $G_X$  be the framed graph of  $G$  with the frame  $X$ , having its framed groupoid  $\mathbb{G}_X$ . Also, let  $(H_{G_X}, L)$  be the canonical framed graph representation of  $\mathbb{G}_X$ , in the sense of Section 4, consisting of the framed graph Hilbert space  $H_{G_X} \stackrel{\text{def}}{=} H_X \otimes H_G$  and the framed groupoid action  $L$  of  $\mathbb{G}_X$ , acting on  $H_{G_X}$ , where  $H_X = L^2(X, \mu)$ , and  $H_G = l^2(\mathbb{G})$ .

In this section, we define a certain operator  $T_{G_X}$  on  $H_{G_X}$ , induced by the framed graph  $G_X$ . The importance of this Hecke-type operator  $T_{G_X}$  is that: it explains how the framed groupoid  $\mathbb{G}_X$  act in an operator algebra  $B(H_{G_X})$ .

In Subsection 3.4, we define the *radial operator*  $T_G$  of  $\mathbb{G}$ , as an element of the graph von Neumann algebra  $M_G = \overline{\mathbb{C}[\mathbb{G}]}^w$  (in  $B(H_G)$ ), by

$$T_G = \sum_{e \in E(G)} (L_e + L_e^*) = \sum_{e \in E(G)} (L_e + L_{e^{-1}}).$$

And we already know, how  $T_G$  acts on the graph Hilbert space  $H_G$  (see [16]), in particular, where  $G$  is a fractal graph. In Subsection 5.3, we introduce the free moment computations  $E(T_G^k)$  of  $T_G$ : if  $G$  is a fractal graph, then

$$E(T_G^k) = |\mathcal{L}_n^o(k)| \cdot 1_{\mathbb{C}^m},$$

where

$$n = \max \{ \deg_{out}(v) : v \in V(G) \} \in \mathbb{N}, \text{ in } G,$$

and

$$m = |V(G)| \in \mathbb{N}_\infty,$$

where  $\mathcal{L}_n^o(k)$  is the collection of all lattice paths in  $\mathbb{R}^2$ , satisfying the axis property.

For any  $B \in \mathcal{B}_X$ , there exists the corresponding characteristic function  $\chi_B$  in  $M_X = L^\infty(X)$ , regarded as a multiplication operator with its symbol  $\chi_B$  on  $H_X = L^2(X)$ . For instance, the identity operator  $1_{M_X}$  of  $M_X$  is identical to  $\chi_X$ . Notice that, as an operator in  $M_X$ , every  $\chi_B$  is a projection, for all  $B \in \mathcal{B}_X$ .

**Assumption.** Without loss of generality, in the rest of this paper, we regard the framed graph von Neumann algebra  $M_{G_X}$  as its \*-isomorphic von Neumann algebra  $\mathcal{M}_{G_X} = M_X \otimes_{\mathbb{C}} M_G$ , i.e., we use  $M_{G_X}$  and  $\mathcal{M}_{G_X}$ , alternatively. □

Now, define the framed radial operators of the framed groupoid  $\mathbb{G}_X$ .

**Definition 7.1.** Let  $G_X$  be a framed graph with its framed groupoid  $\mathbb{G}_X$ , and let  $M_{G_X}$  be the framed graph von Neumann algebra of  $G_X$ .

Define “a” framed radial operator  $T_{G_X}^B$  of  $\mathbb{G}_X$  by the element in  $M_{G_X}$ ,

$$\chi_B \otimes T_G,$$

where  $T_G$  is the radial operator of  $\mathbb{G}$ , in the sense of Subsection 3.4, in the graph von Neumann algebra  $M_G$ , for  $B \in \mathcal{B}_X$ . In particular, this radial operator  $T_{G_X}^B$  is called the *B-framed radial operator* of  $\mathbb{G}_X$ . If  $B = X$ , then we call this *X-framed radial operator*  $T_G^X$ , the full radial operator of  $\mathbb{G}_X$ .

By definition, the *B-framed radial operator*  $T_{G_X}^B$  satisfies

$$T_{G_X}^B \stackrel{\text{def}}{=} \chi_B \otimes T_G$$

$$\begin{aligned}
&= \chi_B \otimes \left( \sum_{e \in E(G)} (L_e + L_{e^{-1}}) \right) \\
&= \sum_{e \in E(G)} (\chi_B \otimes (L_e + L_{e^{-1}})) \\
&= \sum_{e \in E(G)} ((\chi_B \otimes L_e) + (\chi_B \otimes L_{e^{-1}})) \\
&= \sum_{e \in E(G)} ((\chi_B \otimes L_e) + (\chi_B \otimes L_e)^*) \\
&= \sum_{e \in E(G)} (L_{(B,e)} + L_{(B,e)^{-1}}),
\end{aligned}$$

for all  $B \in \mathcal{B}_X$ . Different from the radial operator  $T_G \in M_G$  of the graph groupoid  $\mathbb{G}$ , the framed radial operators  $T_{G_X}^B$  of a framed groupoid  $\mathbb{G}_X$  is also determined by the data, represented by the (measures of) Borel sets  $B$  in  $\mathcal{B}_X$ , too. It is easily shown that, for any singleton sets  $\{t\}$  are contained in  $\mathcal{B}_X$  (if they exist well), then the radial operator  $T_G$  of  $\mathbb{G}$ , and the  $\{t\}$ -framed radial operators  $T_{G_X}^{\{t\}}$  are identically distributed over the diagonal subalgebra  $D_G = \mathbb{C}^{\oplus |V(G)|}$ , for all  $t \in X$ .

Let  $T_{G_X}^B$  be the  $B$ -framed radial operator of  $M_{G_X}$ . The operator  $T_{G_X}^B$  is understood as an amalgamated random variable in the framed  $D_G$ -valued graph  $W^*$ -probability space  $(M_{G_X}, E_X)$  over  $D_G = D_{G_X}$ . So, we can consider the free distributional data of  $T_{G_X}^B$ , for the fixed  $B \in \mathcal{B}_X$ . To do that, we compute the  $D_G$ -valued free moments

$$\{E_X((T_{G_X}^B)^k)\}_{k=1}^\infty \subset D_{G_X} = D_G.$$

In particular, we are interested in the case, where the given framed groupoid  $\mathbb{G}_X$  is a framed fractaloid. By using the free distributional data of the radial operator  $T_G$  of the graph fractaloid  $\mathbb{G}$  (see Subsection 3.2), we can compute the free moments of framed radial operators  $T_{G_X}^B$  of the framed fractaloid  $\mathbb{G}_X$ , for all  $B \in \mathcal{B}_X$ . And, we can realize that the  $k$ -th moment  $E((T_{G_X}^B)^k)$  of  $T_{G_X}^B$  is just a scalar multiplication of  $E(T_G^k)$ , for all  $k \in \mathbb{N}$ .

**Theorem 7.1.** *Let  $T_{G_X}^B \in (M_{G_X}, E_X)$  be the  $B$ -framed radial operator of the framed fractaloid  $\mathbb{G}_X$  of the framed fractal graph  $G_X$ , for  $B \in \mathcal{B}_X$ . Assume that*

$$N = \max\{\text{deg}_{out}(v) : v \in V(G)\}, \text{ in } G.$$

*Then, the  $D_G$ -valued moments of  $T_{G_X}^B$  are determined by*

$$E_X((T_{G_X}^B)^k) = (\mu(B) \cdot |\mathcal{L}_N^o(k)|) \cdot 1_{D_G},$$

*for all  $k \in \mathbb{N}$ .*

**Proof.** Fix  $k \in \mathbb{N}$ . Assume that  $G$  is a fractal graph, satisfying that

$$\text{deg}_{out}(v) = N = \text{deg}_{in}(v), \text{ in } G,$$

for all  $v \in V(G)$ . By definition, the framed graph  $G_X$  of  $G$ , with the frame  $X$ , is a framed fractal graph, and hence, the framed groupoid  $\mathbb{G}_X$  of  $G_X$  is a framed fractaloid. Let  $T_{G_X}^B$  be the  $B$ -framed radial operator, for  $B \in \mathcal{B}_X$ . Then

$$E_X((T_{G_X}^B)^k) = E_X\left(\left(\sum_{e \in E(G)} (L_{(B,e)} + L_{(B,e)}^*)\right)^k\right)$$

$$\begin{aligned}
&= E_X \left( \left( \sum_{e \in E(G)} ((\chi_B \otimes L_e) + (\chi_B \otimes L_{e^{-1}})) \right)^k \right) \\
&= (\mu \otimes E) \left( \left( \chi_B \otimes \left( \sum_{e \in E(G)} (L_e + L_{e^{-1}}) \right) \right)^k \right),
\end{aligned}$$

by the bimodule property of  $\otimes$

$$= (\mu \otimes E) \left( \chi_B^k \otimes \left( \sum_{e \in E(G)} (L_e + L_{e^{-1}}) \right)^k \right).$$

Since

$$\begin{aligned}
L_{(Y, w)}^k &= L_{\left( \frac{Y \cap \dots \cap Y}{k}, w^k \right)} = L_{(Y, w^k)} \\
&\cong \chi_Y \otimes L_{w^k} = \chi_Y^k \otimes L_w^k,
\end{aligned}$$

for all  $(Y, w) \in \mathbb{G}_X$ , and  $k \in \mathbb{N}$ ,

$$\begin{aligned}
&= (\mu(B)) \left( E \left( \left( \sum_{e \in E(G)} (L_e + L_{e^{-1}}) \right)^k \right) \right) \\
&= (\mu(B)) (E(T_G^k)),
\end{aligned}$$

where  $T_G$  is the radial operator of the graph fractaloid  $\mathbb{G}$ , since the radial operator  $T_G$  in the usual graph von Neumann algebra  $M_G$  is defined by

$$T_G = \sum_{e \in E(G)} (L_e + L_{e^{-1}}),$$

i.e., for any  $B \in \mathcal{B}_X$ , a framed radial operator  $T_{G_X}^B$  satisfies

$$E_X((T_{G_X}^B)^k) = (\mu(B))(E(T_G^k)) \text{ in } D_G.$$

Therefore,

$$E_X((T_{G_X}^B)^k) = (\mu(B) \cdot |\mathcal{L}_N^o(k)|) \cdot 1_{D_G},$$

for all  $k \in \mathbb{N}$ , and  $B \in \mathcal{B}_X$ , since  $N$  is the maximum of the out-degrees of vertices of  $G$ , and since

$$E(T_G^k) = |\mathcal{L}_N^o(k)| \cdot 1_{D_G},$$

for all  $k \in \mathbb{N}$  (see Subsection 3.2), where  $\mathcal{L}_N^o(k)$  is the collection of all length- $k$  lattices satisfying the axis property. ■

By the previous theorem, we can obtain the following corollary:

**Corollary 7.2.** *Let  $T_{G_X}^X$  be the full radial operator of the framed fractaloid  $\mathbb{G}_X$ . If the frame  $X = (X, \mathcal{B}_X, \mu)$  of the framed fractal graph  $G_X$  is a Borel probability measure space, in the sense that  $\mu(X) = 1$ , then the operator  $T_{G_X}^X \in M_{G_X}$ , and the radial operator  $T_G \in M_G$  of the graph fractaloid  $\mathbb{G}$  are identically distributed over  $D_G$ .*

**Proof.** By the previous theorem,

$$E_X((T_{G_X}^X)^k) = (\mu(X) \cdot |\mathcal{L}_N^o(k)|) \cdot 1_{D_G},$$

for all  $k \in \mathbb{N}$ . Since the Borel measure space  $X$  is a probability measure space,

$$\mu(X) = 1.$$

Thus,

$$E_X((T_{G_X}^X)^k) = |\mathcal{L}_N^o(k)| \cdot 1_{D_G} = E(T_G^k),$$

for all  $k \in \mathbb{N}$ , under the fractality of  $\mathbb{G}$ . ■

### Appendix: Lattice paths

Let  $\mathbb{R}^2$  be the 2-dimensional  $\mathbb{R}$ -vector space. As usual, we regard  $\mathbb{R}^2$  as the  $\mathbb{R}$ -plane, induced by the horizontal axis (or the  $x$ -axis) and the vertical axis (or the  $y$ -axis). Let  $N \in \mathbb{N}$  be a fixed number. Then, for the given number  $N$ , we define the *lattices*  $l_1, \dots, l_N$  by the vectors in  $\mathbb{R}^2$  by

$$l_k \stackrel{\text{def}}{=} \overrightarrow{(1, e^k)}, \text{ for all } k = 1, \dots, N.$$

To distinguish the point  $(\alpha, \beta)$  in  $\mathbb{R}^2$ , and the vector  $(\alpha, \beta)$ , connecting the origin  $(0, 0)$  to the point  $(\alpha, \beta)$  in  $\mathbb{R}^2$ , we denote the vector  $(\alpha, \beta)$ , by  $\overrightarrow{(\alpha, \beta)}$ , for all  $(\alpha, \beta) \in \mathbb{R}^2$ . Then, the lattices  $l_1, \dots, l_N$  are understood as the upward lattices. Define now the downward lattices  $l_{-1}, \dots, l_{-N}$  by

$$l_{-k} \stackrel{\text{def}}{=} \overrightarrow{(1, -e^k)}, \text{ for all } k = 1, \dots, N.$$

Then, the set  $X_N = \{l_{\pm 1}, \dots, l_{\pm N}\}$  is said to be the (*lattice*) *labelling set*, for  $N \in \mathbb{N}$ . Let  $X_N$  be the labelling set. Then, we can generate *lattice paths* in  $\mathbb{R}^2$  by the following rule: Construct a lattice path  $l_i l_j$ , by transforming the starting point  $(0, 0)$  of  $l_j$  to the ending point  $(1, \varepsilon_i e^i)$  of  $l_i$ , where

$$\varepsilon_i = \begin{cases} 1, & \text{if } i \in \{1, \dots, N\}, \\ -1, & \text{if } i \in \{-1, \dots, -N\}, \end{cases}$$

for all  $i \in \{\pm 1, \dots, \pm N\}$ . By using the iterated attaching (or transforming), we can construct the lattice paths  $l_{i_1} \dots l_{i_n}$ , for  $i_j \in \{\pm 1, \dots, \pm N\}$ , for all  $n \in \mathbb{N}$ . By  $\mathcal{L}_N$ , denote the set of all such lattice paths, generated by the labelling set  $X_N$ , and we call  $\mathcal{L}_N$ , the *lattice path set generated by*  $X_N$ .

Let  $l = l_{i_1} \dots l_{i_n} \in \mathcal{L}_N$ . Then, the length  $|l|$  of  $l$  is defined to be  $n$ , the cardinality of lattices in  $X_N$ , generating the lattice path  $l$ . Define a subset  $\mathcal{L}_N(n)$  of  $\mathcal{L}_N$  by



$$\mathcal{L}_N(n) \stackrel{\text{def}}{=} \{l \in \mathcal{L}_N : |l| = n\},$$

for all  $n \in \mathbb{N}$ . Then, the lattice path set  $\mathcal{L}_N$  is decomposed by

$$\mathcal{L}_N = \bigsqcup_{k=1}^{\infty} \mathcal{L}_N(k),$$

where  $\sqcup$  means the disjoint union. The subsets  $\mathcal{L}_N(k)$  of  $\mathcal{L}_N$  are called the *length- $k$  lattice path set*.

Let  $l \in \mathcal{L}_N$ , and suppose the lattice path  $l$  end on the horizontal axis, in other words, the ending point of the path  $l$  is on the horizontal axis in  $\mathbb{R}^2$ . Then, we say that the lattice path  $l$  *satisfies the (horizontal) axis property*. Define the subsets  $\mathcal{L}_N^o(k)$  of  $\mathcal{L}_N(k)$  by

$$\mathcal{L}_N^o(k) \stackrel{\text{def}}{=} \{l \in \mathcal{L}_N(k) : l \text{ satisfies the axis property}\}.$$

It is easy to check that, by the definition of the lattices  $l_{\pm 1}, \dots, l_{\pm N}$ , the subsets  $\mathcal{L}_N^o(k)$  are empty, whenever  $k$  is odd in  $\mathbb{N}$ . In [14] and [40], we computed the cardinalities of  $\mathcal{L}_N^o(k)$ , for  $N, k \in \mathbb{N}$ .

**Proposition 7.3** (see [14] and [40]). *Let  $N \in \mathbb{N}$ . Then  $|\mathcal{L}_N^o(n)| = 0$ , whenever  $n$  is odd in  $\mathbb{N}$ . For all  $k \in \mathbb{N}$ ,*

$$|\mathcal{L}_N^o(2k)| = \sum_{(j_1, \dots, j_{2k}) \in \mathcal{C}_{2k}} c_{j_1, \dots, j_{2k}},$$

where

$$\mathcal{C}_{2k} = \left\{ (j_1, \dots, j_{2k}) \left| \begin{array}{l} (j_1, \dots, j_{2k}) \in \{\pm 1, \dots, \pm N\}^{2k} \\ j_1 \leq j_2 \leq \dots \leq j_{2k} \\ \sum_{i=1}^{2k} j_i = 0 \end{array} \right. \right\},$$

where the summand  $c_{j_1, \dots, j_{2k}}$  satisfies the following recurrence relation: If there exists  $1 \leq d \leq 2k$  and  $s \in \{1, \dots, 2k\}$  such that

$$c_{j_1, \dots, j_{2k}} = c_{j_1, \dots, j_{s-1}, \underbrace{j_s, j_s, \dots, j_s}_{d\text{-times}}}$$

then it satisfies that

$$c_{j_1, \dots, j_{s-1}, \underbrace{j_s, j_s, \dots, j_s}_{d\text{-times}}} = c_{j_1, \dots, j_{s-1}} \cdot 2^k C_d,$$

with

$$c_{j, j, j, \dots, j} = 1, \text{ for all } j \in \{\pm 1, \dots, \pm N\},$$

where  ${}_m C_n \stackrel{\text{def}}{=} \frac{m!}{n!(m-n)!}$ , for all  $n \leq m \in \mathbb{N}$ . □

For instance, if  $c_{-3, -3, -2, -1, -1, 1, 1, 2, 3, 3} \in \mathcal{C}_{10}$ , then we can compute it by

$$\begin{aligned} & c_{-3, -3, -2, -1, -1, 1, 1, 2, 3, 3} \\ &= (c_{-3, -3, -2, -1, -1, 1, 1, 2})({}_{10}C_2) \\ &= (c_{-3, -3, -2, -1, -1, 1, 1})({}_8C_1)({}_{10}C_2) \\ &= (c_{-3, -3, -2, -1, -1})({}_7C_2)({}_8C_1)({}_{10}C_2) \\ &= (c_{-3, -3, -2})({}_5C_2)({}_7C_2)({}_8C_1)({}_{10}C_2) \\ &= (c_{-3, -3})({}_3C_1)({}_5C_2)({}_7C_2)({}_8C_1)({}_{10}C_2) \\ &= ({}_3C_1)({}_5C_2)({}_7C_2)({}_8C_1)({}_{10}C_2). \end{aligned}$$

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